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COMPARISON ESTIMATES FOR LINEAR FORMS IN ADDITIVE NUMBER THEORY

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ABSTRACT. Let R be a commutative ring R with 1_R and with group of units $R^\times$. Let $\Phi = \Phi(t_1, \ldots, t_h) = \sum_{i=1}^h \varphi_i t_i$ be an h-ary linear form with nonzero coefficients $\varphi_1, \ldots, \varphi_h \in R$. Let M be an R-module. For every subset A of M, the *image of A under Φ* is

$$\Phi(A) = \{\Phi(a_1, \ldots, a_h) : (a_1, \ldots, a_h) \in A^h\}.$$

For every subset I of $\{1, 2, \ldots, h\}$, there is the *subset sum* $s_I = \sum_{i \in I} \varphi_i$. Let $\mathcal{S}(\Phi) = \{s_I : \emptyset \neq I \subseteq \{1, 2, \ldots, h\}\}$.

Theorem. *Let $\Upsilon(t_1, \ldots, t_g) = \sum_{i=1}^g \upsilon_i t_i$ and $\Phi(t_1, \ldots, t_h) = \sum_{i=1}^h \varphi_i t_i$ be linear forms with nonzero coefficients in the ring R. If $\{0, 1\} \subseteq \mathcal{S}(\Upsilon)$ and $\mathcal{S}(\Phi) \subseteq R^\times$, then for every $\varepsilon > 0$ and $c > 1$ there exist a finite R-module M with $|M| > c$ and a subset A of M such that $\Upsilon(A \cup \{0\}) = M$ and $|\Phi(A)| < \varepsilon|M|$.*

1. THE PROBLEM

In 1973, Haight [2] proved that for all positive integers h and ℓ there exist a positive integer m and a subset A of $\mathbf{Z}/m\mathbf{Z}$ such that

$$A - A = \mathbf{Z}/m\mathbf{Z}$$

but the h-fold sumset hA omits ℓ consecutive congruence classes. Ruzsa [5], refining Haight's method, recently proved that, for every positive integer h and every $\varepsilon > 0$, there exist a positive integer m and a subset A of $\mathbf{Z}/m\mathbf{Z}$ such that

$$A - A = \mathbf{Z}/m\mathbf{Z} \qquad \text{and} \qquad |hA| < \varepsilon m.$$

The difference set $A - A$ is the image of A under the linear form $\Upsilon(t_1, t_2) = t_1 - t_2$ and the h-fold sumset hA is the image of A under the linear form $\Phi(t_1, t_2, \ldots, t_h) = t_1 + t_2 + \cdots + t_h$. Equivalently, Ruzsa constructed a subset A of the $\mathbf{Z}$-module $M = \mathbf{Z}/m\mathbf{Z}$ such that

$$\Upsilon(A) = M \qquad \text{and} \qquad |\Phi(A)| < \varepsilon|M|.$$

This is a significant result in additive number theory. In this paper, we extend Ruzsa's theorem to a large class of pairs of linear forms Υ and Φ.

Let R be a commutative ring with multiplicative identity $1_R \neq 0$. We denote the group of units in R by $R^\times$. Associated to every sequence $(\varphi_1, \ldots, \varphi_h)$ of nonzero

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