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Asymptotically trivial linear homogeneous partition inequalities

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ABSTRACT

We consider linear homogeneous partition inequalities of the form

$$\sum_{k=1}^{r} a_k p(n+\mu_k) \leq \sum_{\ell=1}^{s} b_\ell p(n+\nu_\ell)\,, \qquad (*)$$

where $p(n)$ is the number of integer partitions of n, $\{a_1, a_2, \cdots, a_r\}$, $\{b_1, b_2, \cdots, b_s\}$ are positive integers, and $0 \leq \mu_1 < \mu_2 < \cdots < \mu_r$, $0 \leq \nu_1 < \nu_2 < \cdots < \nu_s$ are integers. From the fact that $\lim_{n\to\infty} \frac{p(n+\mu)}{p(n)} = 1$ (μ an integer) it follows that the inequality $(*)$ can only hold if $\sum_{k=1}^{r} a_k \leq \sum_{\ell=1}^{s} b_\ell$. If the last relation is a strict inequality than $(*)$ holds for all $n > N$, for an appropriately specified N, and can be established for all $n \geq 1$ by verifying that it holds for the finite set of cases specified by $1 \leq n \leq N$. Such inequalities will be referred to as *asymptotically trivial*. Several examples of such inequalities are presented. The inequality $(*)$ is *trivial* if the stronger condition $\sum_{k=1}^{r} a_k p(\mu_k - \min(\mu_1, \nu_1) + 1) \leq \sum_{\ell=1}^{s} b_\ell$ holds, *i.e.*, the supremum of the left-hand side of $(*)$ is smaller than or equal to the infimum of its right-hand side. If $\sum_{k=1}^{r} a_k = \sum_{\ell=1}^{s} b_\ell$ then we say that $(*)$ is *non-trivial*. In this case $(*)$ can be an identity. A "conventional" proof, establishing the nature of $(*)$ for all n, is required.

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1. Introduction

In the present paper we consider linear homogeneous partition inequalities of the form

$$\sum_{k=1}^{r} a_k p(n+\mu_k) \leq \sum_{\ell=1}^{s} b_\ell p(n+\nu_\ell)\,, \tag{1}$$

where $p(n)$ is the number of integer partitions of n, $\{a_1, a_2, \cdots, a_r\}$, $\{b_1, b_2, \cdots, b_s\}$ are positive integers, and $0 \leq \mu_1 < \mu_2 < \cdots < \mu_r$, $0 \leq \nu_1 < \nu_2 < \cdots < \nu_s$ are integers. From the fact that $\lim_{n\to\infty} \frac{p(n+\mu)}{p(n)} = 1$ (μ an integer) it follows that the inequality (1) can only hold if $\sum_{k=1}^{r} a_k \leq \sum_{\ell=1}^{s} b_\ell$. If the last relation is a strict inequality than (1) holds for all $n > N$, for an appropriately specified N, and can be established for all $n \geq 1$ by verifying that it holds for all $1 \leq n \leq N$, a task which is easily feasible for at least $N \sim 10000$. Such inequalities will be referred to as *asymptotically trivial*. Several examples of such inequalities are presented. An argument of this type was recently used by Bessenrodt and Ono [3]. Admittedly, proofs of the kind described above do not provide the insight that a "conventional" proof often yields. However, they provide a fairly powerful generator of inequalities, motivating the search for more insightful "conventional" proofs. The inequality (1) is *trivial* if the stronger condition $\sum_{k=1}^{r} a_k p(\mu_k - \min(\mu_1, \nu_1)+1) \leq \sum_{\ell=1}^{s} b_\ell$ holds, *i.e.*, the supremum of the left-hand side of (1) is smaller than or equal to the infimum of its right-hand side. Finally, if $\sum_{k=1}^{r} a_k = \sum_{\ell=1}^{s} b_\ell$ then we say that (1) is *non-trivial*. In this case (1) can be an identity. A "conventional" proof, establishing the nature of (1) for all n, is required.

The characterization of the three classes of linear partition inequalities, *non-trivial*, *asymptotically trivial* and *trivial*, is the main contribution of the present article. These three classes are illustrated by the inequalities

$$p(n) + p(n-5) \leq p(n-1) + p(n-2)\,, \tag{2}$$

$$p(n) \leq p(n-1) + p(n-2)\,, \tag{3}$$

and

$$p(n) \leq 2p(n-1)\,, \tag{4}$$

respectively. Inequality (2) is *non-trivial* in the sense defined above (the sum of the coefficients on the left-hand side is equal to the sum of the coefficients on the right-hand side, so nothing can be said about the validity of this inequality by examining the asymptotic behavior of the partition numbers). It was proved by Merca [15] using Euler's generating function for $p(n)$, an approach that can certainly be extended to many more linear partition inequalities. Inequality (2) was shown by Andrews and Merca [2] to be a member of an infinite family of partition inequalities, all of which are non-trivial in the sense defined above. While inequality (3) follows straightforwardly from inequality (2), it can be proved independently by noting that it is *asymptotically trivial* in the sense defined

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