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On the class number divisibility of pairs of imaginary quadratic fields

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ABSTRACT

We construct an infinite family of pairs of imaginary quadratic fields $\mathbb{Q}(\sqrt{D})$ and $\mathbb{Q}(\sqrt{D+1})$ with $D \in \mathbb{Z}$ whose class numbers are both divisible by 3.

1. Introduction

Let l be the prime 3, 5, or 7 and let m and n integers with $m \neq 0$. Iizuka, Konomi, and Nakano [3] constructed an infinite family of pairs of quadratic fields $\mathbb{Q}(\sqrt{D})$ and $\mathbb{Q}(\sqrt{mD+n})$ with $D \in \mathbb{Q}$ whose class numbers are both divisible by l. For the case $n = 0$, it is easy to see that D can be retaken in $\mathbb{Z}$. See Komatsu [5] and Iizuka, Konomi, and Nakano [2] (see also Komatsu [6] for a recent related result). Whereas, when $n \neq 0$,

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it is essential to distinguish whether D is an integer or not. The aim of the present paper is to show that D can be taken to be an integer in the above result for pairs of imaginary quadratic fields when $l = 3$ and $m = n = 1$. Our main result is the following theorem:

Theorem 1. *There is an infinite family of pairs of imaginary quadratic fields* $\mathbb{Q}(\sqrt{D})$ *and* $\mathbb{Q}(\sqrt{D+1})$ *with* $D \in \mathbb{Z}$ *whose class numbers are both divisible by* 3.

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2. Proof of Theorem 1

Let $f(X) \in \mathbb{Z}[X]$ be an irreducible cubic polynomial and M the splitting field of $f(X)$ over $\mathbb{Q}$. We denote by $D(f)$ the discriminant of $f(X)$. Suppose that $D(f)$ is not a square and put $k = \mathbb{Q}(\sqrt{D(f)})$. Then $M/\mathbb{Q}$ is an S_3-extension, k is a quadratic field, and M/k is a cyclic extension of degree 3. If M/k is unramified, the class number of k is divisible by 3. A prime ideal of k above a prime number p is ramified in M if and only if p is totally ramified in the cubic field generated by a root of $f(X)$ (see, for example, Kishi and Miyake [4]).

For a prime number p and an integer n, we denote by $v_p(n)$ the greatest exponent m such that $p^m \mid n$. The following lemma follows immediately from Theorem 1 in Llorente and Nart [7]:

Lemma 1. *Suppose that the cubic polynomial*

$$f(X) = X^3 - aX - b, \quad a, b \in \mathbb{Z}$$

is irreducible over $\mathbb{Q}$ *and that either* $v_q(a) < 2$ *or* $v_q(b) < 3$ *holds for every prime number* q. *Let* M *be the splitting field of* $f(X)$ *over* $\mathbb{Q}$. *We denote by* $D(f)$ *the discriminant of* $f(X)$. *Suppose that* $D(f)$ *is not a square and put* $k = \mathbb{Q}(\sqrt{D(f)})$. *Let* p *be a prime number and* $\mathfrak{p}$ *a prime ideal of* k *above* p.

(i) *Case* $p \neq 3$. *The prime ideal* $\mathfrak{p}$ *is unramified in* M *if and only if one of the following conditions holds:*
  (a) $v_p(a) = 0$.
  (b) $v_p(b) = 0$.
  (c) $1 = v_p(a) < v_p(b)$.

(ii) *Case* $p = 3$. *If* $3 \mid a$ *and* $3 \nmid b$, *then* $\mathfrak{p}$ *is unramified in* M *if and only if one of the following conditions holds:*
  (a) $a \equiv 0,\ b \equiv \pm 1 \pmod 9$.
  (b) $a \equiv 6,\ b \equiv \pm 4 \pmod 9$.
  (c) $a \equiv 3,\ b \equiv \pm 2 \pmod{27}$.
  (d) $a \equiv 12,\ b \equiv \pm 11 \pmod{27}$.
  (e) $a \equiv 21,\ b \equiv \pm 7 \pmod{27}$.