ARTICLE IN PRESS YJNTH:5845

Journal of Number Theory ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

An analytic heuristic for multiplicity computation for Zaremba's Conjecture

Peter Cohen

Massachusetts Institute of Technology, United States

ARTICLE INFO

Article history:
Received 24 October 2016
Received in revised form 17 August 2017
Accepted 18 August 2017
Available online xxxx
Communicated by S.J. Miller

Keywords:
Zaremba's Conjecture
Hardy–Littlewood Circle Method
Continued fractions

ABSTRACT

Zaremba's Conjecture concerns the formation of continued fractions with partial quotients restricted to a given alphabet. In order to answer the numerous questions that arrive from this conjecture, it is best to consider a semi-group, often denoted Γ_A, which arises naturally as a subset of $SL_2(\mathbb{Z})$ when considering finite continued fractions. To translate back from this semi-group into rational numbers, we select a projection mapping satisfying certain criteria to recover the numerator and denominator of the continued fractions in rational form. The central question of our research is to determine the multiplicity of a given denominator. To this end, we develop a heuristic method similar to the Hardy–Littlewood Circle Method. We compare this theoretical model to the exact data, gleaned by simulation, and demonstrate that our formula appears to be asymptotically valid. We then evaluate different aspects of the accuracy of our formula.

Contents

E-mail address: plcohen@mit.edu.

https://doi.org/10.1016/j.jnt.2017.08.016

1. Introduction

For any real number $\alpha \in [0,1]$ we may write α as a continued fraction of the form

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} \tag{1}$$

In this notation, each a_i is called a partial quotient, and we will denote α by $[a_1, \ldots]$. We restrict the possible values of a_i to be in some alphabet $\mathcal{A} \subseteq \mathbb{N}$. It is a well known fact that rational numbers have finite length continued fractions which are unique if restricted to an even number of partial quotients.

Zaremba's Conjecture [Zar72] states that there exists $A \in \mathbb{N}$ such that for all $q \in \mathbb{N}$ there exists $a \leq q$ that is co-prime to q where $\frac{a}{q}$ has partial quotients bounded by A.[1] Thus, in the case of Zaremba's conjecture, the alphabet $\mathcal{A}$ is simply the set $\{1, \ldots, A\}$. To study this conjecture, we rely upon the observation that, for $\frac{b}{d} = [a_1, \ldots, a_n]$

$$\begin{bmatrix} * & b \\ * & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & a_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & a_2 \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ 1 & a_n \end{bmatrix}. \tag{2}$$

Thus, it is natural to consider the set of matrices

$$S = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & i \end{bmatrix} \right\}_{i=1}^{A}. \tag{3}$$

This set can then be used to form all finite length continued fractions with partial quotients within $\mathcal{A} = \{1, \cdots, A\}$ by forming

$$\Gamma_A = \langle S \rangle^+ \cap SL_2(\mathbb{Z}), \tag{4}$$

where $\langle S \rangle^+$ denotes the semigroup generated by S.

Notice that the restriction imposed by intersecting $\langle S \rangle^+$ with $SL_2(\mathbb{Z})$ causes all elements of Γ_A to be of an even number of partial quotients; however, as noted before,

[1] It should be noted that Zaremba's Conjecture has been proven for a density 1 subset of $\mathbb{N}$ by Kontorovich and Bourgain in [BK14].

Please cite this article in press as: P. Cohen, An analytic heuristic for multiplicity computation for Zaremba's Conjecture, J. Number Theory (2018), https://doi.org/10.1016/j.jnt.2017.08.016