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The number of tagged parts over the partitions with designated summands

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ABSTRACT

We are concerned with two types of partitions considered by Andrews, Lewis and Lovejoy. One is the partitions with designated summands where exactly one is tagged among parts with equal size. The other is the partitions with designated summands where all parts are odd. In this paper, we study two partition functions $\mathrm{PD_t}(n)$ and $\mathrm{PDO_t}(n)$, which count the number of tagged parts over the above two types of partitions respectively. We first give the generating functions of $\mathrm{PD_t}(n)$ and $\mathrm{PDO_t}(n)$. Then we establish many congruences modulo small powers of 3 for them. Finally, we pose some problems for future work.

1. Introduction

Andrews, Lewis and Lovejoy [2] investigated the number $\mathrm{PD}(n)$ of partitions of n with designated summands. A partition with designated summands is obtained from an ordinary partition by tagging exactly one of each part size. They also considered $\mathrm{PDO}(n)$, the number of partitions of n with designated summands in which all parts are odd. For example, the following is a list of all the desired partitions of 5:

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$$5', 4' + 1', 3' + 2', 3' + 1' + 1, 3' + 1 + 1', 2' + 2 + 1', 2 + 2' + 1',$$
$$2' + 1' + 1 + 1, 2' + 1 + 1' + 1, 2' + 1 + 1 + 1', 1' + 1 + 1 + 1 + 1,$$
$$1 + 1' + 1 + 1 + 1, 1 + 1 + 1' + 1 + 1, 1 + 1 + 1 + 1' + 1, 1 + 1 + 1 + 1 + 1'.$$

Thus, $\mathrm{PD}(5) = 15$ and $\mathrm{PDO}(5) = 8$.

Andrews, Lewis and Lovejoy [2] derived the generating functions of $\mathrm{PD}(n)$ and $\mathrm{PDO}(n)$:

$$\sum_{n=0}^{\infty} \mathrm{PD}(n) q^n = \frac{f_6}{f_1 f_2 f_3},$$
$$\sum_{n=0}^{\infty} \mathrm{PDO}(n) q^n = \frac{f_4 f_6^2}{f_1 f_3 f_{12}},$$

where $f_k = (q^k; q^k)_\infty$ and

$$(a; q)_\infty = \prod_{n=1}^{\infty} (1 - a q^{n-1}).$$

By examining q-series identities, they found a Ramanujan type congruence for $\mathrm{PD}(n)$ and determined the parity of $\mathrm{PDO}(n)$. They further applied modular forms to obtain the generating functions for $\mathrm{PD}(n)$ and $\mathrm{PDO}(n)$ in certain arithmetic progressions.

In his work on generalized divisor sums, MacMahon [15] considered $A_{n,k}$ which in fact is the number of partitions of n with designated summands wherein exactly k different magnitudes occur among all the parts. For more results on $A_{n,k}$, see Andrews and Rose [3]. Inspired by their work, we introduce the partition function $\mathrm{PD_t}(n)$, which counts the total number of tagged parts over all the partitions of n with designated summands. For instance, $\mathrm{PD_t}(5) = 24$. We will show that the generating function of $\mathrm{PD_t}(n)$ satisfies

$$\sum_{n=0}^{\infty} \mathrm{PD_t}(n) q^n = \frac{1}{2} \left(\frac{f_3^5}{f_1^3 f_6^2} - \frac{f_6}{f_1 f_2 f_3} \right). \tag{1.1}$$

Based on this, we can prove several congruences modulo 3 and 9 for $\mathrm{PD_t}(n)$. For example, for $n \geq 0$,

$$\mathrm{PD_t}(3n + 2) \equiv 0 \pmod{3}, \tag{1.2}$$
$$\mathrm{PD_t}(36n + 21) \equiv 0 \pmod{9}, \tag{1.3}$$
$$\mathrm{PD_t}(36n + 33) \equiv 0 \pmod{9}. \tag{1.4}$$

We shall also study the partition function $\mathrm{PDO_t}(n)$, which enumerates the number of tagged parts over all the partitions of n with designated summands in which all parts

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