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Ranks of overpartitions modulo 6 and 10

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Lovejoy and Osburn proved formulas for the generating functions for the rank differences of overpartitions modulo 3 and 5. In this paper, we derive formulas for the generating functions for the rank differences of overpartitions modulo 6 and 10. With these generating functions, we obtain some equalities and inequalities on ranks of overpartitions modulo 6 and 10. We also relate these generating functions to the third order mock theta functions $\omega(q)$ and $\rho(q)$ and the tenth order mock theta functions $\phi(q)$ and $\psi(q)$.

1. Introduction

The rank of a partition was introduced by Dyson [11] as the largest part of the partition minus the number of parts. He [11] conjectured that this partition statistic provided

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combinatorial interpretations of Ramanujan's congruences $p(5n+4) \equiv 0 \pmod 5$ and $p(7n+5) \equiv 0 \pmod 7$, where $p(n)$ is the number of partitions of n. More precisely, let $N(m,n)$ denote the number of partitions of n with rank m and let $N(s,\ell,n)$ denote the number of partitions of n with rank congruent to s modulo ℓ. Dyson conjectured

$$N(k,5,5n+4) = \frac{p(5n+4)}{5}, \quad 0 \le k \le 4, \tag{1.1}$$

$$N(k,7,7n+5) = \frac{p(7n+5)}{7}, \quad 0 \le k \le 6. \tag{1.2}$$

These two assertions were confirmed by Atkin and Swinnerton-Dyer [5]. In fact, they established generating functions for the rank differences $N(s,\ell,\ell n+d) - N(t,\ell,\ell n+d)$ with $\ell = 5$ or 7 and for $0 \le d,s,t < \ell$, many of which are in terms of infinite products and generalized Lambert series. Although Dyson's rank fails to explain Ramanujan's congruence $p(11n+6) \equiv 0 \pmod{11}$ combinatorially, the generating functions for the rank differences $N(s,\ell,\ell n+d) - N(t,\ell,\ell n+d)$ with $\ell = 11$ have also been determined by Atkin and Hussain [4]. Since then, the rank differences of partitions modulo other numbers have been extensively studied. For example, Lewis established the rank differences of partitions modulo 2 in [20] and the rank differences of partitions modulo 9 in [19]. Santa-Gadea [29] obtained other rank differences of partitions modulo 9 and some rank differences of partitions modulo 12. Recently, Mao [25] established the generating functions for the rank differences of partitions modulo 10.

Dyson's rank can be extended to overpartitions in the obvious way. Recall that an overpartition [10] is a partition in which the first occurrence of a part may be overlined. The rank of an overpartition is defined to be the largest part of an overpartition minus its number of parts. Similarly, let $\overline{N}(m,n)$ denote the number of overpartitions of n with rank m, and let $\overline{N}(s,\ell,n)$ denote the number of overpartitions of n with rank congruent to s modulo ℓ. Lovejoy [21] obtained the following generating function for $\overline{N}(m,n)$,

$$\overline{R}(z;q) := \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \overline{N}(m,n) z^m q^n = \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+n}}{(1-zq^n)(1-z^{-1}q^n)}. \tag{1.3}$$

Analogous to the rank of a partition, Lovejoy and Osburn [22] studied the rank differences $\overline{N}(s,\ell,\ell n+d) - \overline{N}(t,\ell,\ell n+d)$ with $\ell = 3$ or 5 for $0 \le d,s,t < \ell$. The rank differences with $\ell = 7$ have been recently determined by Jennings-Shaffer [18]. It has been shown in [9] that there are no congruences of the form $\overline{p}(\ell n+d) \equiv 0 \pmod{\ell}$ for primes $\ell \ge 3$. The generating functions for these rank differences provide a measure of the extent to which the rank fails to produce a congruence $\overline{p}(\ell n+d) \equiv 0 \pmod{\ell}$. On the other hand, as remarked by Jennings-Shaffer in [18], determining these three difference formulas is equivalent to determining the 3-dissection of $\overline{R}(\exp(2i\pi/3);q)$, the 5-dissection of $\overline{R}(\exp(2i\pi/5);q)$ and the 7-dissection of $\overline{R}(\exp(2i\pi/7);q)$.

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