ARTICLE IN PRESS YJNTH:5852

Journal of Number Theory ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Zeros of the first derivative of Dirichlet L-functions

Hirotaka Akatsuka [a,*], Ade Irma Suriajaya [b,1]

[a] *Otaru University of Commerce, 3-5-21, Midori, Otaru, Hokkaido, 047-8501, Japan*
[b] *Nagoya University, Furo-cho, Chikusa-ku, Nagoya, Aichi, 464-8602, Japan*

A R T I C L E I N F O

Article history:
Received 22 January 2017
Received in revised form 26 July 2017
Accepted 18 August 2017
Available online xxxx
Communicated by the Principal Editors

MSC:
11M06

Keywords:
Dirichlet L-function
Derivative
Zeros

A B S T R A C T

Yıldırım has classified zeros of the derivatives of Dirichlet L-functions into trivial zeros, nontrivial zeros and vagrant zeros. In this paper we remove the possibility of vagrant zeros for $L'(s,\chi)$ when the conductors are large to some extent. Then we improve asymptotic formulas for the number of zeros of $L'(s,\chi)$ in $\{s \in \mathbb{C} : \mathrm{Re}(s) > 0, |\mathrm{Im}(s)| \leq T\}$. We also establish analogues of Speiser's theorem, which characterize the generalized Riemann hypothesis for $L(s,\chi)$ in terms of zeros of $L'(s,\chi)$, when the conductor is large.

1. Introduction

Let χ be a primitive Dirichlet character modulo $q > 1$. We denote the Dirichlet L-function attached to χ by $L(s,\chi)$. In this paper we investigate zeros of $L'(s,\chi)$, the first derivative of $L(s,\chi)$. Previously, Yıldırım [8] has investigated zeros of the k-th

* Corresponding author.
E-mail addresses: akatsuka@res.otaru-uc.ac.jp (H. Akatsuka), m12026a@math.nagoya-u.ac.jp, adeirmasuriajaya@riken.jp (A.I. Suriajaya).

[1] Current address: Interdisciplinary Mathematical Sciences Team, RIKEN, 2-1, Hirosawa, Wako, Saitama, 351-0198, Japan.

https://doi.org/10.1016/j.jnt.2017.08.023

derivative $L^{(k)}(s,\chi)$ of $L(s,\chi)$ for any given $k \in \mathbb{Z}_{\geq 1}$. He has shown that $L^{(k)}(s,\chi)$ does not vanish when $\mathrm{Re}(s)$ is sufficiently large. He has also obtained a zero-free region in a left half-plane. Strictly speaking, for any fixed $\varepsilon > 0$ there exists $K > 0$, which depends only on k and ε, such that $L^{(k)}(s,\chi)$ has no zeros in $\{s = \sigma + it : |s| > q^K, \sigma < -\varepsilon, |t| > \varepsilon\}$. Based on the above facts, he classified zeros of $L^{(k)}(s,\chi)$ in the following way (see [8, §7]):

- trivial zeros, which are located on $\{\sigma + it : \sigma \leq -q^K, |t| \leq \varepsilon\}$,
- vagrant zeros, which are located on $\{s = \sigma + it : |s| \leq q^K, \sigma \leq -\varepsilon\}$,
- nontrivial zeros, which are located in $\{\sigma + it : \sigma > -\varepsilon\}$.

Loosely speaking, one of our main results is to remove the possibility of vagrant zeros for $L'(s,\chi)$. In order to state it precisely, we put

$$\Theta(\chi) := \sup\{\mathrm{Re}(\rho) : \rho \in \mathbb{C}, L(\rho,\chi) = 0\}.$$

It is easy to check that the following properties hold:

- $1/2 \leq \Theta(\chi) \leq 1$.
- $\Theta(\overline{\chi}) = \Theta(\chi)$.
- For each primitive Dirichlet character χ, the generalized Riemann hypothesis (GRH, in short) for $L(s,\chi)$ is equivalent to $\Theta(\chi) = 1/2$.

The following theorem says that $L'(s,\chi)$ does not vanish apart from $\mathrm{Im}(s) = 0$ on a left half-plane:

Theorem 1.1. *Let χ be a primitive Dirichlet character modulo $q > 1$. Then $L'(s,\chi)$ has no zeros on $s \in \mathcal{D}_1(\chi) \cup \mathcal{D}_2(\chi)$, where*

$$\mathcal{D}_1(\chi) = \left\{\sigma + it : \sigma \leq 1 - \Theta(\chi),\ |t| \geq \frac{6}{\log q}\right\} \setminus \{\rho \in \mathbb{C} : L(\rho,\chi) = 0\},$$
$$\mathcal{D}_2(\chi) = \left\{\sigma + it : \sigma \leq -q^2,\ |t| \geq \frac{12}{\log|\sigma|}\right\}.$$

Remark. The constants 6 and 12 in $\mathcal{D}_1(\chi)$ and $\mathcal{D}_2(\chi)$ can be replaced by smaller constants. Along with our method proving Theorem 1.1, we expect that $\mathcal{D}_1(\chi)$ and $\mathcal{D}_2(\chi)$ are replaced by

$$\left\{\sigma + it : \sigma \leq 1 - \Theta(\chi),\ |t| \geq \frac{c}{\log q}\right\} \setminus \{\rho \in \mathbb{C} : L(\rho,\chi) = 0\}, \tag{1.1}$$

$$\left\{\sigma + it : \sigma \leq -1,\ |t| \geq \frac{c}{\log(q|\sigma|)}\right\}, \tag{1.2}$$

respectively, where c is a little larger than π (see also Remark following the proof of Theorem 1.1 in §2). However, if we wish to improve Theorem 1.1, it may be better to use an information on the number of zeros of $L'(s,\chi)$ on certain strips, which is stated

Please cite this article in press as: H. Akatsuka, A.I. Suriajaya, Zeros of the first derivative of Dirichlet L-functions, J. Number Theory (2018), https://doi.org/10.1016/j.jnt.2017.08.023