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Algebraic independence of the values of functions satisfying Mahler type functional equations under the transformation represented by a power relatively prime to the characteristic of the base field

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ABSTRACT

We give positive characteristic analogues of complex entire functions having remarkable property that their values as well as their derivatives of any order at any nonzero algebraic numbers are algebraically independent. These results are obtained by establishing a criterion for the algebraic independence of the values of Mahler functions as well as that of the algebraic independence of the Mahler functions themselves over any function fields of positive characteristic.

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1. Introduction and the results

Some complex entire functions are known to have the property that their values as well as their derivatives of any order at any nonzero algebraic numbers are algebraically independent. One of the examples of such functions is $F(x) := \sum_{k=0}^{\infty} \alpha^{d^k} x^k$, where d is an integer greater than 1 and α is an algebraic number with $0 < |\alpha| < 1$. Nishioka [10] proved that the infinite subset $\{F^{(j)}(a) \mid j \geq 0,\ a \in \overline{\mathbb{Q}}^{\times}\}$ of $\mathbb{C}$ consisting of the values of $F(x)$ and its derivatives of any order is algebraically independent over the field $\mathbb{Q}$ of rational numbers, where $\overline{\mathbb{Q}}^{\times}$ is the set of nonzero algebraic numbers. This result was obtained by using the fact that the function $f(x,z) := \sum_{k=0}^{\infty} x^k z^{d^k}$, for which $F(x) = f(x,\alpha)$, satisfies

$$f(x,z) = xf(x,z^d) + z. \tag{1}$$

Mahler functions are those which satisfy the functional equations such as (1) or (4) below and the arithmetic properties of their complex values have been investigated by various authors. Mahler functions are also studied over base field of positive characteristic.

Let $\mathbb{F}_q$ be a field with $q = p^e$ elements, where p is a prime and $e > 0$ is an integer. We regard the principal ideal domain $A := \mathbb{F}_q[\theta]$ of polynomials of the indeterminate θ with coefficients in $\mathbb{F}_q$ as an analogy of the ring $\mathbb{Z}$ of rational integers. Then the field of fractions $K := \mathbb{F}_q(\theta)$ of A is an analogy of $\mathbb{Q}$. In this paper we consider the algebraic independence over K of the values of Mahler functions. In considering the positive characteristic analogue of complex values, we treat the completions of the field K with respect to the absolute values defined by (2) and (3) below.

The well-defined map $v_\infty : K \to \mathbb{Z} \cup \{\infty\}$ given by $v_\infty(0) := \infty$ and, for any $a = b/c$ with b and c nonzero elements of A, by

$$v_\infty(a) := -\deg_\theta a = \deg_\theta c - \deg_\theta b,$$

where the right-hand side does not depend on the choice of b and c, is a *valuation* on K. We define the normalized absolute value $|\cdot|_\infty$ corresponding to v_∞ by

$$|a|_\infty := q^{-v_\infty(a)} = q^{\deg_\theta a} \quad (a \in K^\times), \qquad |0|_\infty := 0. \tag{2}$$

Moreover, we denote by v_P the valuation determined by the maximal ideal P of A and define the normalized absolute value $|\cdot|_P$ corresponding to v_P by

$$|a|_P := \left(q^{\deg_\theta f}\right)^{-v_P(a)} = q^{-\deg_\theta f \cdot \mathrm{ord}_f a} \quad (a \in K^\times), \qquad |0|_P := 0, \tag{3}$$

where f is the monic irreducible polynomial which generates the principal ideal P. Then the normalized absolute values defined by (2) and (3) satisfy the product formula (6) in the next section. We denote by $|\cdot|_v$ the absolute value corresponding to the valuation v, where v is one of v_∞ and v_P defined above, and by K_v the completion of K with

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