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On depth 2 zeta-like families ☆

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Multizeta values for $\mathbb{F}_q[\theta]$ were initially studied by Thakur, who defined them as analogues of classical multiple zeta values of Euler. In this present paper we give certain depth 2 families of zeta-like multizeta values, namely those whose ratio to the zeta value of the same weight is rational.

1. Introduction

The study of arithmetic of zeta values begins by Euler's famous evaluations: for $m \in \mathbb{N}$,

$$\zeta(2m) = \frac{-B_{2m}\left(2\pi\sqrt{-1}\right)^{2m}}{2(2m)!},$$

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where $B_{2m} \in \mathbb{Q}$ are Bernoulli numbers. Euler's formula implies that $\zeta(n)/(2\pi\sqrt{-1})^n$ is rational if and only if n is even. The multiple zeta values (abbreviate MZVs) $\zeta(s_1, \cdots, s_r)$, where $s_1, \ldots, s_r$ are positive integers with $s_1 \geq 2$, are generalizations of zeta values. Here r is called the depth and $w := \sum_{i=1}^{r} s_i$ is called the weight of $\zeta(s_1, \ldots, s_r)$. These numbers were first studied by Euler for the case of $r = 2$. He derived certain relations between zeta and multiple zeta values, such as $\zeta(2, 1) = \zeta(3)$. We call $\zeta(s_1, \ldots, s_r)$ *Eulerian* (*zeta-like* respectively) if the ratio $\zeta(s_1, \ldots, s_r)/\big(2\pi\sqrt{-1}\big)^w$ $(\zeta(s_1, \ldots, s_r)/\zeta(w)$ respectively) is rational. It seems difficult to determine whether any given MZV is zeta-like or non-zeta-like. Brown [2, Theorem 3.3] gave a sufficient condition for zeta-like MZV's using the theory of motivic multiple zeta values.

Let $A = \mathbb{F}_q[\theta]$ be the polynomial ring in one variable over the finite field $\mathbb{F}_q$ with quotient field K and p be the characteristic of K. Carlitz introduced and derived an analogue of Euler's formula for what we now called Carlitz zeta values $\zeta_A(n)$. Let $\mathbf{C}$ be the Carlitz module and $\tilde{\pi} := (-\theta)^{\frac{q}{q-1}} \prod_{i=1}^{\infty} \left(1 - \frac{\theta}{\theta^{q^i}}\right)^{-1}$ be a fundamental period of $\mathbf{C}$, where $(-\theta)^{\frac{1}{q-1}}$ is a fixed choice of $(q-1)$-th root of $-\theta$. Carlitz [3] derived an analogue of Euler's formula for $\zeta_A(n)$. He showed that if $(q-1) \mid n$, then

$$\zeta_A(n) = \frac{\mathrm{BC}(n)}{\Gamma_{n+1}} \tilde{\pi}^n, \tag{1.0.1}$$

where $\mathrm{BC}(n) \in K$ are the Bernoulli–Carlitz numbers and $\Gamma_{n+1} \in A$ are the Carlitz factorials. Here n is called *even* if $q - 1 \mid n$. Note that $\tilde{\pi}^n \in \mathbb{F}_q((\frac{1}{\theta}))$ if and only if n is *even*, and so Carlitz's result implies that $\zeta_A(n)/\tilde{\pi}^n \in K$ if and only if n is *even*.

Thakur introduced the multizeta values $\zeta_A(s_1, \ldots, s_r)$: for any r-tuple $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$,

$$\zeta_A(\mathfrak{s}) = \zeta_A(s_1, \ldots, s_r) := \sum \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in \mathbb{F}_q((\frac{1}{\theta})),$$

where the sum is over $(a_1, \ldots, a_r) \in A_+^r$ with $\deg_\theta a_1 > \cdots > \deg_\theta a_r$. We still refer MZV's for these values. Here r is called the *depth* and $w := s_1 + \cdots + s_r$ is called the *weight* of $\zeta_A(\mathfrak{s})$. Thakur [11,12] showed that these MZV's are non-vanishing and gave interesting relations between MZV's. As in the classical case, Thakur called $\zeta_A(s_1, \ldots, s_r)$ *Eulerian* if $\zeta_A(s_1, \ldots, s_r)/\tilde{\pi}^w \in K$. On the other hand, Thakur called $\zeta_A(s_1, \ldots, s_r)$ *zeta-like* if $\zeta_A(s_1, \ldots, s_r)/\zeta_A(w) \in K$, where w is the weight of $\zeta_A(s_1, \ldots, s_r)$. We mention that in fact, the ratio $\zeta_A(s_1, \ldots, s_r)/\zeta_A(w)$ is known to be algebraic over K if and only if it is rational in K (see [4]). We further mention that Eulerian MZV's only occur in *even* weights (see [5]), and hence Eulerian MZV's are zeta-like by Carlitz's formula.

It is a natural question to ask how to find all Eulerian/zeta-like MZV's. Lara Rodriguez and Thakur [9] proved precise formulas for certain families of Eulerian/zeta-like MZV's, and gave conjectures on which r-tuples $(s_1, \ldots, s_r)$ may occur for zeta-like MZV's. Their conjectures are supported by numerical data from continued fraction computations.