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Hecke structures of weakly holomorphic modular forms and their algebraic properties ☆

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Let $S_k^!(\Gamma_1(N))$ be the space of weakly holomorphic cusp forms of weight k on $\Gamma_1(N)$ with an even integer $k > 2$ and $M_k^!(\Gamma_1(N))$ be the space of weakly holomorphic modular forms of weight k on $\Gamma_1(N)$. Further, let z denote a complex variable and $D := \frac{1}{2\pi i}\frac{\partial}{\partial z}$. In this paper, we construct a basis of the space $S_k^!(\Gamma_1(N))/D^{k-1}(M_{2-k}^!(\Gamma_1(N)))$ consisting of Hecke eigenforms by using the Eichler–Shimura cohomology theory. Further, we study algebraicity of CM values of weakly holomorphic modular forms in the basis. This applies to an analogue of the Chowla–Selberg formula for a mock modular form whose shadow is the Ramanujan delta function.

1. Introduction

Let N be a positive integer. Let $S_k(\Gamma_1(N))$ be the space of cusp forms of weight k on $\Gamma_1(N)$. For a positive integer n, we have the Hecke operator T_n acting on the

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space $S_k(\Gamma_1(N))$. The Hecke operators play a significant role in the study of the structure of modular forms. The action of the Hecke operators can be extended to the space $M_k^!(\Gamma_1(N))$ of weakly holomorphic modular forms of weight k on $\Gamma_1(N)$, and recently the structures of Hecke operators on $M_k^!(\Gamma_1(N))$ were investigated by some researchers (for example, see [3,7,8,13]). In this paper, we construct a basis of the space $S_k^!(\Gamma_1(N))/D^{k-1}(M_{2-k}^!(\Gamma_1(N)))$ consisting of Hecke eigenforms by using the Eichler–Shimura cohomology theory. Here, $S_k^!(\Gamma_1(N))$ denotes the space of weakly holomorphic cusp forms of weight k on $\Gamma_1(N)$; a weakly holomorphic cusp form means a weakly holomorphic modular form whose constant terms vanish at all cusps. Further, we study the algebraicity of CM values of weakly holomorphic modular forms in the basis. This applies to an analogue of the Chowla–Selberg formula for a mock modular form whose shadow is the Ramanujan delta function.

Throughout the paper, Γ_1 denotes $\mathrm{SL}_2(\mathbb{Z})$ and k is a positive even integer. We set $\Gamma = \Gamma_1(N)$ with a positive integer N. Let $w = 2-k$ and let V_w be the module of complex polynomials of degree at most w with Γ_1-action given by the usual slash operator $|_{-w}$. Let $\tilde{V}_w^\Gamma$ be the induced Γ_1-module $\mathrm{Ind}_\Gamma^{\Gamma_1}(V_w)$. Then, by the Shapiro lemma, we have an isomorphism between parabolic cohomology groups (see [25, Section 2])

$$H_P^1(\Gamma, V_w) \cong H_P^1(\Gamma_1, \tilde{V}_w^\Gamma).$$

Let $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and $U = TS$. We define

$$W_w^\Gamma := \left\{ P \in V_w^\Gamma \;\middle|\; P + P|_{-w}S = 0,\ P + P|_{-w}U + P|_{-w}U^2 = 0 \right\}, \tag{1.1}$$
$$C_w^\Gamma := \left\{ P - P|_{-w}S \;\middle|\; P \in V_w^\Gamma,\ P|_{-w}T = P \right\},$$

where V_w^Γ is the subspace of $\tilde{V}_w^\Gamma$ defined by

$$V_w^\Gamma := \left\{ P \in \tilde{V}_w^\Gamma \;\middle|\; P|_{-w}(-I) = P \right\}, \tag{1.2}$$

where I denotes the identity matrix in Γ_1. Then, we have an isomorphism

$$H_P^1(\Gamma_1, \tilde{V}_w^\Gamma) \cong W_w^\Gamma / C_w^\Gamma.$$

From this isomorphism, we obtain the following theorem, which is a generalization of the result of Bringmann, Guerzhoy, Kent, and Ono [3, Theorem 1.2].

Theorem 1.1. *Let k be a positive integer with $k \geq 2$ and $\Gamma = \Gamma_1(N)$. Then we have an exact sequence*

$$0 \longrightarrow M_{2-k}^!(\Gamma) \xrightarrow{D^{k-1}} S_k^!(\Gamma) \xrightarrow{r_k} W_w^\Gamma / C_w^\Gamma \longrightarrow 0, \tag{1.3}$$