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Lehmer pairs and derivatives of Hardy's Z-function

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Occurrences of very close zeros of the Riemann zeta function are strongly connected with Lehmer pairs and with the Riemann Hypothesis. The aim of the present note is to derive a condition for a pair of consecutive simple zeros of the ζ-function to be a Lehmer pair in terms of derivatives of Hardy's Z-function. Furthermore, we connect Newman's conjecture with stationary points of the Z-function, and present some numerical results.

1. Introduction

Hardy's Z-function is defined by:

$$Z(t) := e^{\mathrm{i}\vartheta(t)}\zeta\left(\frac{1}{2}+\mathrm{i}t\right), \quad \vartheta(t) := \frac{1}{2\mathrm{i}}\log\frac{\Gamma\left(\frac{1}{4}+\mathrm{i}\frac{t}{2}\right)}{\Gamma\left(\frac{1}{4}-\mathrm{i}\frac{t}{2}\right)} - \frac{\log\pi}{2}t.$$

In 1956, D. H. Lehmer found a pair of very close zeros of the Z-function. Visually this means that the graph $Z(t)$ barely crosses the t-axis at these zeros, and this poses a threat

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to the validity of the Riemann Hypothesis, see [Edw01]. The occurrence of such pairs of close zeros is known as *Lehmer's phenomenon* and pairs are called *Lehmer pairs*. The paper [CSV94] gives a precise meaning to the notion of Lehmer pairs and proves a result which is today's most promising method to find lower bounds λ of the *de Bruijn–Newman constant* Λ. It should be noted that the Riemann Hypothesis is equivalent to $\Lambda \le 0$ and *Newman's conjecture* is $\Lambda \ge 0$, see [New76, Theorem 3 and Remark 2].

Let $\{\gamma_1, \gamma_2\}$ be a pair of two simple zeros of the Z-function and define

$$\bar{g}_{\{\gamma_1,\gamma_2\}} := (\gamma_1 - \gamma_2)^2 \sum_{\gamma \notin \{\gamma_1,\gamma_2\}} \frac{1}{(\gamma - \gamma_1)^2} + \frac{1}{(\gamma - \gamma_2)^2} \tag{1}$$

where γ goes through all zeros of the Z-function. Assuming the Riemann Hypothesis, a pair $\{\gamma_-, \gamma_+\}$ of two consecutive simple zeros of the Z-function is said to be a Lehmer pair if $\bar{g}_{\{\gamma_-,\gamma_+\}} < 4/5$. If this is the case, then

$$\lambda_{\{\gamma_-,\gamma_+\}} := \frac{(\gamma_+ - \gamma_-)^2}{2\bar{g}_{\{\gamma_-,\gamma_+\}}} \left(\left(1 - \frac{5}{4} \bar{g}_{\{\gamma_-,\gamma_+\}} \right)^{\frac{4}{5}} - 1 \right) \le \Lambda$$

and an infinite number of Lehmer pairs implies $\Lambda \ge 0$, see [CSV94, Theorem 1 and Corollary 1]. Because the bound $4/5$ is the least restrictive possible that allows the proof of the above inequality to go through, it is not surprising that Lehmer pairs are not so rare; the first thousand zeros contain 48 Lehmer pairs.

In order to produce an "analytic" definition of Lehmer pair, Stopple gives in [Sto17, Theorem 1] a more restrictive definition in terms of the first three derivatives of the Riemann xi-function $\Xi(t)$ at the pair's zeros. In this note we extend his result to derivatives of the Z-function.

Theorem 1. *Let $Z(t)$ be Hardy's Z-function and define the real function $\widehat{F}$ by*

$$\widehat{F}(t) := -\frac{Z'''}{Z'}(t) + \frac{3}{4} \left(\frac{Z''}{Z'} \right)^2 (t). \tag{2}$$

Let $\{\gamma_1, \gamma_2\}$ be a pair of two simple zeros of the Z-function and define

$$\hat{g}_{\{\gamma_1,\gamma_2\}} := \frac{1}{3} (\gamma_1 - \gamma_2)^2 \left(\widehat{F}(\gamma_1) + \widehat{F}(\gamma_2) \right) - 2.$$

Under the Riemann Hypothesis we have

$$0 < \bar{g}_{\{\gamma_1,\gamma_2\}} - \hat{g}_{\{\gamma_1,\gamma_2\}} < 3 (\gamma_1 - \gamma_2)^2 \left(\frac{1}{\gamma_1^2} + \frac{1}{\gamma_2^2} \right) < 3\hat{g}_{\{\gamma_1,\gamma_2\}}. \tag{3}$$

This estimate is obviously very good for consecutive and relatively large zeros. The following immediate corollary gives conditions for a pair of zeros to be a Lehmer pair.

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