Journal of Number Theory 183 (2018) 1–23

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Explicit estimates for the distribution of numbers free of large prime factors

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ARTICLE INFO

Article history:
Received 5 May 2017
Accepted 4 August 2017
Available online 25 September 2017
Communicated by S.J. Miller

Keywords:
Smooth numbers
Friable numbers
Saddle point method

ABSTRACT

There is a large literature on the asymptotic distribution of numbers free of large prime factors, so-called *smooth* or *friable* numbers. But there is very little known about this distribution that is numerically explicit. In this paper we follow the general plan for the saddle point argument of Hildebrand and Tenenbaum, giving explicit and fairly tight intervals in which the true count lies. We give two numerical examples of our method, and with the larger one, our interval is so tight we can exclude the famous Dickman–de Bruijn asymptotic estimate as too small and the Hildebrand–Tenenbaum main term as too large.

1. Introduction

For a positive integer $n > 1$, denote by $P(n)$ the largest prime factor of n, and let $P(1) = 1$. Let $\Psi(x, y)$ denote the number of $n \le x$ with $P(n) \le y$. Such integers n are known as y-smooth, or y-friable. Asymptotic estimates for $\Psi(x, y)$ are quite useful

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https://doi.org/10.1016/j.jnt.2017.08.039
0022-314X/

in many applications, not least of which is in the analysis of factorization and discrete logarithm algorithms.

One of the earliest results is due to Dickman [6] in 1930, who gave an asymptotic formula for $\Psi(x,y)$ in the case that x is a fixed power of y. Dickman showed that

$$\Psi(x,y) \sim x\rho(u) \qquad (y\to\infty,\ x=y^u) \tag{1.1}$$

for every fixed $u \ge 1$, where $\rho(u)$ is the "Dickman–de Bruijn" function, defined to be the continuous solution of the delay differential equation

$$\begin{aligned} u\rho'(u)+\rho(u-1)&=0 \qquad (u>1),\\ \rho(u)&=1 \qquad (0\le u\le 1). \end{aligned}$$

There remain the questions of the error in the approximation (1.1), and also the case when $u = \log x/\log y$ is allowed to grow with x and y. In 1951, de Bruijn [3] proved that

$$\Psi(x,y) = x\rho(u)\Big(1+O_\varepsilon\Big(\frac{\log(1+u)}{\log y}\Big)\Big)$$

holds uniformly for $x \ge 2$, $\exp\{(\log x)^{5/8+\varepsilon}\} < y \le x$, for any fixed $\varepsilon > 0$. After improvements in the range of this result by Maier and Hensley, Hildebrand [10] showed that the de Bruijn estimate holds when $\exp(\{(\log\log x)^{5/3+\varepsilon}\}) \le y \le x$.

In 1986, Hildebrand and Tenenbaum [11] provided a uniform estimate for $\Psi(x,y)$ for all $x \ge y \ge 2$, yielding an asymptotic formula when y and u tend to infinity. The starting point for their method is an elementary argument of Rankin [17] from 1938, commonly known now as Rankin's "trick". For complex s, define

$$\zeta(s,y) = \sum_{\substack{n\ge 1\\ P(n)\le y}} n^{-s} = \prod_{p\le y}(1-p^{-s})^{-1}$$

(where p runs over primes) as the partial Euler product of the Riemann zeta function $\zeta(s)$. In the case that $s=\sigma$ is real and $0<\sigma<1$, we have

$$\Psi(x,y) = \sum_{\substack{n\le x\\ P(n)\le y}} 1 \le \sum_{P(n)\le y} (x/n)^\sigma = x^\sigma\zeta(\sigma,y). \tag{1.2}$$

Then σ can be chosen optimally to minimize $x^\sigma\zeta(\sigma,y)$.

Let

$$\phi_j(s,y) = \frac{\partial^j}{\partial s^j}\log\zeta(s,y).$$

The function