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Mean values of derivatives of L-functions in function fields: II

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ABSTRACT

This is the second part of our investigation of mean values of derivatives of L-functions in function fields. In this paper, specifically, we prove several mean values results for the derivatives of L-functions in function fields when the average is taken over all discriminants, i.e., over all monic polynomials of a prescribed degree in $\mathbb{F}_q[T]$. We establish exact formulas for the mean value of the μ-th derivative of L-functions in function fields at the critical point and we compute a few particular examples.

1. Introduction

In the first paper in this series, see [1], we computed the first moment of derivatives of quadratic Dirichlet L-functions in function fields using classical techniques such as character sums, approximate functional equation and the Riemann Hypothesis for

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curves. In particular, we calculated the full polynomial in the asymptotic expansion of $\sum_{D\in\mathcal{H}} L''(\frac{1}{2},\chi_D)$, with $L(s,\chi_D)$ being the quadratic Dirichlet L-function attached to the quadratic character χ_D where D is monic and square-free and $\mathcal{H}$ is the set of all monic and square-free polynomials of odd degree in $\mathbb{F}_q[T]$. In a subsequent paper, we will show how to generalize the result above and compute the first moment of the μ-th derivative of quadratic Dirichlet L-functions and formulate conjectures for higher moments of derivatives of this family of L-functions.

In this paper, we prove several mean values results for the derivatives of L-functions in function fields when the average is taken over all discriminants, i.e., over all monic polynomials of a prescribed degree. Differently from the first paper, where we computed the first moment for the second derivative, here we are going to compute the first moment for the generic μ-th derivative for different families of L-functions, where $\mu \geq 1$.

This series of papers has two main motivations. The first motivation comes from the study of moments of derivatives of the Riemann zeta function, which was initiated by Ingham [8], and then further developed by the work of Conrey [2], Gonek [5], Conrey, Rubinstein and Snaith [3], Hughes, Keating and O'Connell [7], Milinovich and Ng [11, 12], Laurinčikas and Steuding [10] and several other authors. The second motivation is coming from the pioneering work of Hoffstein and Rosen [6] about the study of mean values of Dirichlet L-functions in function fields. See [1] for a summarized account of some of the previous results about moments of derivatives of the Riemann zeta function.

Before we proceed and state the main results of this paper and some previous results on mean values of L-functions in function fields we are going to need a few basic definitions about the theory of L-functions in function fields. See [13] for a detailed exposition about L-functions in function fields.

Let $\mathbb{F}_q$ be a finite field of odd cardinality and we denote $A = \mathbb{F}_q[T]$ to be the polynomial ring over $\mathbb{F}_q$ and $k = \mathbb{F}_q(T)$ to be the rational function field over $\mathbb{F}_q$. The norm of a polynomial $f \in A$ is defined to be $|f| = q^{\deg(f)}$. We now introduce the zeta function of A to be

$$\begin{aligned}\zeta_A(s) &= \sum_{f \text{ monic}} |f|^{-s} \\ &= \frac{1}{1-q^{1-s}}.\end{aligned} \tag{1.1}$$

From now on sums over polynomials will denote sums over monic polynomials unless the contrary is stated.

For m a non-square in A we define the Dirichlet character modulo m using the Jacobi symbol in $\mathbb{F}_q[T]$ by

$$\chi_m(f) = \left(\frac{m}{f}\right), \tag{1.2}$$