Journal of Number Theory 183 (2018) 62–73

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

An upper bound for the number of solutions of ternary purely exponential diophantine equations ☆

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A R T I C L E   I N F O

Article history:
Received 24 May 2016
Received in revised form 12 July 2017
Accepted 12 July 2017
Available online 25 August 2017
Communicated by the Principal Editors

MSC:
11D61
11J86

Keywords:
Ternary purely exponential diophantine equation
Upper bound for solution number
Application of Gel'fond–Baker method

A B S T R A C T

Let a, b, c be fixed coprime positive integers with $\min\{a, b, c\} > 1$. In this paper, combining the Gel'fond–Baker method with an elementary approach, we prove that if $\max\{a, b, c\} > 5 \times 10^{27}$, then the equation $a^x + b^y = c^z$ has at most three positive integer solutions (x, y, z).

☆ Supported by the National Natural Science Foundation of China (No. 10971184).
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http://dx.doi.org/10.1016/j.jnt.2017.07.004
0022-314X/

1. Introduction

Let $\mathbb{N}$ be the set of all positive integers. Let a, b, c be fixed coprime positive integers with $\min\{a, b, c\} > 1$. In 1933, K. Mahler [13] used his p-adic analogue of the Thue–Siegel method to prove that the ternary purely exponential diophantine equation

$$a^x + b^y = c^z, \quad x, \quad y, \quad z \in \mathbb{N} \tag{1.1}$$

has only finitely many solutions (x, y, z). His method is ineffective. An effective result for solutions of (1.1) was given by A.O. Gel'fond [7]. Let $N(a, b, c)$ denote the number of solutions (x, y, z) of (1.1). As a straightforward consequence of an upper bound for the number of solutions of binary S-unit equations due to F. Beukers and H.P. Schlickewei [2], we have $N(a, b, c) \leq 2^{36}$. In recent years, many papers investigated the exact values of $N(a, b, c)$. The known results showed that (1.1) has only a few solutions for some special cases (see [5], [6], [11], [12], [14], [15], [16], [17], [18], [19], [20] and [21]). Very recently, the authors [8] proved that if a, b, c satisfy certain divisibility conditions and $\max\{a, b, c\}$ is large enough, then (1.1) has at most one solution (x, y, z) with $\min\{x, y, z\} > 1$. In this paper we prove a general result as follows:

Theorem 1.1. *If* $\max\{a, b, c\} > 5 \times 10^{27}$*, then* $N(a, b, c) \leq 3$.

Notice that if $(a, b, c) = (3, 5, 2)$, then (1.1) has exactly three solutions $(x, y, z) = (1, 1, 3), (3, 1, 5)$ and $(1, 3, 7)$. Perhaps, in general, $N(a, b, c) \leq 3$ is the best upper bound for $N(a, b, c)$.

2. An upper bound for the solutions of (1.1)

In [8], combining a lower bound for linear forms in two logarithms and an upper bound for the p-adic logarithms due to M. Laurent [9] and Y. Bugeaud [3] respectively, the authors proved that all solutions (x, y, z) of (1.1) satisfy $\max\{x, y, z\} < 155000(\log \max\{a, b, c\})^3$, where log is used for natural logarithm. In this section, using the same method as in [8], we make a slight improvement as follows:

Theorem 2.1. *All solutions* (x, y, z) *of* (1.1) *satisfy*

$$\max\{x, y, z\} < 6500(\log \max\{a, b, c\})^3. \tag{2.1}$$

The proof of Theorem 2.1 depends on the following lemmas.

Lemma 2.1. ([10], Corollaire 2 et Tableau 2) *Let* $\alpha_1, \alpha_2, \beta_1, \beta_2$ *be positive integers with* $\min\{\alpha_1, \alpha_2\} \geq 2$. *Further let* $\Lambda = \beta_1 \log \alpha_1 - \beta_2 \log \alpha_2$. *If* $\Lambda \neq 0$*, then*

$$\log |\Lambda| > -32.31(\log \alpha_1)(\log \alpha_2)(\max\{10, 0.18 + \log(\frac{\beta_1}{\log \alpha_2} + \frac{\beta_2}{\log \alpha_1})\})^2.$$