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$\mathcal{I}_c^{(q)}$-convergence of arithmetical functions [☆]

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ABSTRACT

Let $n > 1$ be an integer with its canonical representation, $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Put $H(n) = \max\{\alpha_1, \dots, \alpha_k\}$, $h(n) = \min\{\alpha_1, \dots, \alpha_k\}$, $\omega(n) = k$, $\Omega(n) = \alpha_1 + \cdots + \alpha_k$, $f(n) = \prod_{d|n} d$ and $f^*(n) = \frac{f(n)}{n}$. Many authors deal with the statistical convergence of these arithmetical functions. For instance the notion of normal order is defined by means of statistical convergence. The statistical convergence is equivalent with $\mathcal{I}_d$-convergence, where $\mathcal{I}_d$ is the ideal of all subsets of positive integers having the asymptotic density zero. In this paper we will study $\mathcal{I}$-convergence of well known arithmetical functions, where $\mathcal{I} = \mathcal{I}_c^{(q)} = \{A \subseteq \mathbb{N} : \sum_{a \in A} a^{-q} < +\infty\}$ is an admissible ideal on $\mathbb{N}$ for $q \in (0, 1\rangle$ such that $\mathcal{I}_c^{(q)} \subsetneq \mathcal{I}_d$.

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1. Introduction

The notion of statistical convergence was introduced in [6], [24] and the notion of $\mathcal{I}$-convergence from the paper [16] corresponds to the natural generalization of statistical convergence (see also [4] where $\mathcal{I}$-convergence is defined by means of filter—the dual notion to ideal). These notions have been developed in several directions in [2], [3], [5], [9], [14], [15], [19], [22] and have been used in various parts of mathematics, in particular in number theory and ergodic theory, for example [1], [7], [10], [11], [13], [15], [20], [21], [23]. Recall the definition and some examples of ideals on $\mathbb{N}$.

Let $\mathcal{I} \subseteq 2^{\mathbb{N}}$. $\mathcal{I}$ is called an admissible ideal of subsets of positive integers, if $\mathcal{I}$ is additive (if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$), hereditary (if $A \in \mathcal{I}$ and $B \subset A$ then $B \in \mathcal{I}$), containing all singletons and it does not contain $\mathbb{N}$. Here we present some examples of admissible ideals.

More examples can be found in the papers [11], [14] and [18].

Example 1.

a) The class of all finite subsets of $\mathbb{N}$ forms an admissible ideal usually denoted by $\mathcal{I}_f$.
b) Let ϱ be a density function on $\mathbb{N}$, the set $\mathcal{I}_\varrho = \{A \subseteq \mathbb{N} \ : \ \varrho(A) = 0\}$ is an admissible ideal. We will use namely the ideals $\mathcal{I}_d$, $\mathcal{I}_\delta$, $\mathcal{I}_u$ and $\mathcal{I}_h$ related to asymptotic, logarithmic, uniform and Alexander density respectively. For the definitions of these densities see e.g. [1], [8], [11], [14], [18] and [25].
c) For an $q \in (0, 1\rangle$ the set $\mathcal{I}_c^{(q)} = \{A \subseteq \mathbb{N} \ : \ \sum_{a\in A} a^{-q} < +\infty\}$ is an admissible ideal. The ideal $\mathcal{I}_c^{(1)} = \{A \subseteq \mathbb{N} \ : \ \sum_{a\in A} a^{-1} < +\infty\}$ is usually denoted by $\mathcal{I}_c$. It is easy to see, that for any $q_1, q_2 \in (0, 1)$, $q_1 < q_2$ we have

$$\mathcal{I}_f \subsetneq \mathcal{I}_c^{(q_1)} \subsetneq \mathcal{I}_c^{(q_2)} \subsetneq \mathcal{I}_c \subsetneq \mathcal{I}_d \subsetneq \mathcal{I}_\delta. \tag{1}$$

d) Let $\mathbb{N} = \bigcup_{j=1}^{\infty} D_j$ be a decomposition on $\mathbb{N}$ (i.e. $D_k \cap D_l = \emptyset$ for $k \neq l$). Assume that D_j $(j = 1, 2, \dots)$ are infinite sets (e.g. we can choose $D_j = \{2^{j-1}.(2s-1) \ : \ s \in \mathbb{N}\}$ for $j = 1, 2, \dots$). Denote $\mathcal{I}_{\mathbb{N}}$ the class of all $A \subseteq \mathbb{N}$ such that A intersects only a finite number of D_j. Then $\mathcal{I}_{\mathbb{N}}$ is an admissible ideal.

Let us recall notions of $\mathcal{I}$- and $\mathcal{I}^*$-convergence of sequences of real numbers see [16].

Definition 2.

(i) We say that a sequence $x = (x_n)_{n=1}^{\infty}$ $\mathcal{I}$-converges to a number L and we write $\mathcal{I} - \lim x_n = L$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \ : \ |x_n - L| \geq \varepsilon\}$ belongs to the ideal $\mathcal{I}$.
(ii) Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$. A sequence $x = (x_n)_{n=1}^{\infty}$ of real numbers is said to be $\mathcal{I}^*$-convergent to $L \in \mathbb{R}$, if there is a set $H \in \mathcal{I}$, such that for $M = \mathbb{N} \setminus H = \{m_1 < m_2 < \cdots\}$ we have