ARTICLE IN PRESS YJNTH:5817

Journal of Number Theory ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

The growth of class numbers of quadratic Diophantine equations

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ARTICLE INFO

Article history:
Received 3 March 2017
Received in revised form 29 July 2017
Accepted 29 July 2017
Available online xxxx
Communicated by the Principal Editors

MSC:
11E12
11E41

Keywords:
Growth of class number

ABSTRACT

Let F be a totally real algebraic number field, with O_F the ring of algebraic integers in F, let L be an O_F-lattice on a d-dimensional ($d \geq 2$) positive definite quadratic space V over F and let u_0 be a primitive vector in L. The main objective of this paper is to study when a is a fixed (non-unit) algebraic integer in O_F and n is a positive rational integer, how the class numbers of lattices translation $L + \frac{u_0}{a^n}$ grow as n tends to infinity.

1. Introduction

In the arithmetical theory of quadratic forms, the notion of class number can be viewed as a measure of the obstruction of the local–global principal for the classical representation problem of quadratic forms. Then the computation of class numbers is an important problem in this theory. The mass formulae introduced by Minkowski in [2] and in a better form by Siegel in [5], [6] and [7] give a method for computing the class number of quadratic forms.

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http://dx.doi.org/10.1016/j.jnt.2017.07.009
0022-314X/© 2017 Published by Elsevier Inc.

In [8], we reformulate Minkowski's mass formula for quadratic Diophantine equations. As an example, we apply this mass formula to calculate class numbers of quadratic Diophantine equations. In this paper, we apply this mass formula to study the growth of class numbers of certain quadratic Diophantine equations.

Throughout the paper, F will be a totally real algebraic number field with ring of algebraic integers O_F. Let Ω_F be the set of all non-trivial primes on F. F_p will stand for the completion of F at a prime $p \in \Omega_F$. Denote ∞_F as the set of all archimedean primes on F and $p < \infty_F$ for $p \in \Omega_F \backslash \infty_F$. If $p < \infty_F$ we let O_{F_p} stand for the valuation ring of F_p and $O_{F_p}^\times$ for the group of units. We also let $q_p = |O_{F_p}/(\pi_p)|$, the letter π_p denotes a choice of uniformizer at $p < \infty_F$, and we let ord_p refer to the discrete exponential order function with $\mathrm{ord}_p(\pi_p) = 1$.

The subsequent discussion will be conducted in the geometric language of quadratic spaces and lattices, and any unexplained notations and terminologies can be found in [3].

Let V be a non-degenerate quadratic space over F with the symmetric bilinear map

$$B : V \times V \longrightarrow F \text{ with } q(x) = B(x, x)$$

for any $x \in V$ and the special orthogonal group

$$SO(V) = \{\sigma \in GL(V) : q(\sigma x) = q(x) \text{ for any } x \in V \text{and } \det\sigma = 1\}.$$

Let $SO_{\mathbf{A}}(V)$ and $GL_{\mathbf{A}}(V)$ be the adele group of $SO(V)$ and $GL(V)$ respectively.

A lattice L on V is defined as a finitely generated O_F-module in V satisfying $FL = V$. We denote the localization of a lattice L in the localization V_p of V at p by L_p for $p < \infty_F$. We note that if $p \in \infty_F$, then $L_p = V_p$.

Consider a lattice L on V. For any non-zero vector u_0 in V we define the lattice translation in V to be the set $L + u_0$. We can always realize a quadratic Diophantine equation as being induced from a free lattice translation in some quadratic space (see section 1 of [8]).

From Lemma 1.2 of [8], we can define the action of $SO_{\mathbf{A}}(V)$ on $L + u_0$.

Definition 1.1. Let $L + u_0$ be a lattice translation on V. Define

$$\mathrm{gen}^+(L + u_0) = \text{the orbit of } L + u_0 \text{ under the action of } SO_{\mathbf{A}}(V)$$

called the genus of $L + u_0$ and

$$\mathrm{cls}^+(L + u_0) = \text{the orbit of } L + u_0 \text{ under the action of } SO(V)$$

called the class of $L + u_0$.

The number of classes in $\mathrm{gen}^+(L + u_0)$ is called the class number of $L + u_0$.

Please cite this article in press as: L. Sun, The growth of class numbers of quadratic Diophantine equations, J. Number Theory (2017), http://dx.doi.org/10.1016/j.jnt.2017.07.009