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Arithmetic properties of Delannoy numbers and Schröder numbers ☆

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Define

$$D_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 x^k (x+1)^{n-k} \quad \text{for } n = 0, 1, 2, \ldots$$

and

$$s_n(x) = \sum_{k=1}^{n} \frac{1}{n} \binom{n}{k} \binom{n}{k-1} x^{k-1} (x+1)^{n-k} \quad \text{for } n = 1, 2, 3, \ldots.$$

Then $D_n(1)$ is the n-th central Delannoy number D_n, and $s_n(1)$ is the n-th little Schröder number s_n. In this paper we obtain some surprising arithmetic properties of $D_n(x)$ and $s_n(x)$. We show that

$$\frac{1}{n} \sum_{k=0}^{n-1} D_k(x) s_{k+1}(x) \in \mathbb{Z}[x(x+1)] \quad \text{for all } n = 1, 2, 3, \ldots.$$

Moreover, for any odd prime p and p-adic integer $x \not\equiv 0, -1 \pmod{p}$, we establish the supercongruence

$$\sum_{k=0}^{p-1} D_k(x) s_{k+1}(x) \equiv 0 \pmod{p^2}.$$

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As an application we confirm Conjecture 5.5 in [S14a], in particular we prove that

$$\frac{1}{n}\sum_{k=0}^{n-1} T_k M_k (-3)^{n-1-k} \in \mathbb{Z} \quad \text{for all } n = 1, 2, 3, \ldots,$$

where T_k is the k-th central trinomial coefficient and M_k is the k-th Motzkin number.

1. Introduction

For $m, n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, the Delannoy number

$$D_{m,n} := \sum_{k\in\mathbb{N}} \binom{m}{k}\binom{n}{k} 2^k \tag{1.1}$$

in combinatorics counts lattice paths from $(0, 0)$ to (m, n) in which only east $(1, 0)$, north $(0, 1)$, and northeast $(1, 1)$ steps are allowed (cf. R.P. Stanley [St99, p. 185]). The n-th central Delannoy number $D_n = D_{n,n}$ has another well-known expression:

$$D_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} = \sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k}. \tag{1.2}$$

For $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, the n-th little Schröder number is given by

$$s_n := \sum_{k=1}^{n} N(n, k) 2^{k-1} \tag{1.3}$$

with the Narayana number $N(n, k)$ defined by

$$N(n, k) := \frac{1}{n}\binom{n}{k}\binom{n}{k-1} \in \mathbb{Z}.$$

(See [Gr, pp. 268–281] for certain combinatorial interpretations of the Narayana number $N(n, k) = N(n, n + 1 - k)$.) For $n \in \mathbb{N}$, the n-th large Schröder number is given by

$$S_n := \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\frac{1}{k+1} = \sum_{k=0}^{n} \binom{n+k}{2k} C_k, \tag{1.4}$$

where C_k denotes the Catalan number $\binom{2k}{k}/(k+1) = \binom{2k}{k} - \binom{2k}{k+1}$. It is well known that $S_n = 2s_n$ for every $n = 1, 2, 3, \ldots$. Both little Schröder numbers and large Schröder numbers have many combinatorial interpretations (cf. [St97] and [St99, pp. 178, 239–240]);