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On interlacing of the zeros of a certain family of modular forms

Ekata Saha *, N. Saradha

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Navy Nagar, Mumbai, 400 005, India

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For $s \in \{0, 4, 6, 8, 10, 14\}$, let $k = 12m(k) + s \geq 12$ be an even integer and f_k be a normalised modular form of weight k with real Fourier coefficients, written as

$$f_k = E_k + \sum_{j=1}^{m(k)} a_j^{(k)} E_{k-12j} \Delta^j.$$

Under suitable conditions on $a_j^{(k)}$ (rectifying an earlier result of Getz), we show that all the zeros of f_k, in the standard fundamental domain for the action of $\mathbf{SL}(2, \mathbb{Z})$ on the upper half plane, lie on the arc $A := \{e^{i\theta} : \pi/2 \leq \theta \leq 2\pi/3\}$. Further, we provide a criterion for a family of normalised modular forms $\{f_k\}_k$ so that the zeros of f_k and f_{k+12} interlace on $A^\circ := \{e^{i\theta} : \pi/2 < \theta < 2\pi/3\}$.

1. Introduction

Let $\mathbb{H}$ denote the complex upper half plane. The full modular group $\mathbf{SL}(2, \mathbb{Z})$ acts on $\mathbb{H}$ by the transformation law

* Corresponding author.
E-mail addresses: ekata@math.tifr.res.in (E. Saha), saradha@math.tifr.res.in (N. Saradha).

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$$z \mapsto \frac{az+b}{cz+d}$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbf{SL}(2,\mathbb{Z})$. The standard fundamental domain for this action of $\mathbf{SL}(2,\mathbb{Z})$ on $\mathbb{H}$ is the following subset of $\mathbb{H}$,

$$\mathbb{F} := \left\{ |z| \geq 1, -\frac{1}{2} \leq \Re(z) \leq 0 \right\} \cup \left\{ |z| > 1, 0 < \Re(z) < \frac{1}{2} \right\}.$$

Throughout this article we take $k \geq 4$ to be an even integer. For $z \in \mathbb{H}$, the Eisenstein series of weight k for the full modular group $\mathbf{SL}(2,\mathbb{Z})$ is defined by the following absolutely convergent series,

$$E_k(z) := \frac{1}{2} \sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)=1}} \frac{1}{(cz+d)^k}.$$

The Eisenstein series of weight 0 is defined by $E_0 := 1$. The Eisenstein series E_k is a modular form of weight k for $\mathbf{SL}(2,\mathbb{Z})$. It is classical that the space of modular forms of weight k is generated by the Eisenstein series E_k and cusp forms of weight k. We write $k = 12m(k) + s$ with $s \in \{0, 4, 6, 8, 10, 14\}$. We will use this notation for k throughout the article without any further mention. The unique normalised cusp form of weight 12, denoted by Δ, is defined as follows:

$$\Delta := \frac{E_4^3 - E_6^2}{1728}.$$

Rankin and Swinnerton-Dyer [8] proved that for $k \geq 4$, all the zeros of the Eisenstein series E_k lie in the arc

$$A := \left\{ |z| = 1, -\frac{1}{2} \leq \Re(z) \leq 0 \right\} = \left\{ e^{i\theta} : \frac{\pi}{2} \leq \theta \leq \frac{2\pi}{3} \right\}.$$

In 2004, extending the arguments of Rankin and Swinnerton-Dyer, Getz [4, Theorem 1] gave a criterion for a normalised modular form f of weight k for $\mathbf{SL}(2,\mathbb{Z})$, written as $f = E_k + \sum_{j=1}^{m(k)} a_j E_{k-12j}\Delta^j$, to have all its zeros on the arc A, in terms of the a_j's.

However, there is a rectifiable error in his proof. While estimating $H(\theta) := e^{ik\theta/2} f(e^{i\theta})$ for $\theta \in [\pi/2, 2\pi/3]$ (see [4, p. 2225, eq. (2.5)]), he used an upper bound for R_{k-12j} from [4, p. 2224, eq. (2.3)] for each $1 \leq j \leq m(k)$, where $R_{k-12j} := e^{i(k-12j)\theta/2} E_{k-12j}(e^{i\theta}) - 2\cos((k-12j)\theta/2)$. But this upper bound is valid if $k - 12j \geq 12$. Note that $k - 12m(k)$ is always less than 12, unless it is 14. We present a corrected version of his theorem below. For this we setup the following notation.

For $k = 0$ or $k \geq 4$ an even integer, let the real number δ_k be defined as follows: