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Orders of Quaternion Algebras with Involution

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Abstract

We introduce the notion of maximal orders over quaternion algebras with orthogonal involution and give a classification over local and global fields. Over local fields, we show that there is a correspondence between maximal and/or modular lattices and orders closed under the involution.

1. Introduction:

The study of maximal orders of quaternion algebras has a rich history, going back to the work of Hurwitz in the 1800s. Finding such orders is essentially a solved problem. Indeed, an explicit algorithm is given in [1] that constructs maximal orders in a semisimple algebra over an algebraic number field in polynomial time, with a simpler algorithm given in [2] for the special case of quaternion algebras. It is a straight-forward exercise to check that any order of a quaternion algebra is closed under the standard involution, i.e. quaternion conjugation. However, given an arbitrary involution ‡ on a quaternion algebra H and an order $\mathcal{O} \subset H$, it need not be the case that $\mathcal{O} = \mathcal{O}^{\ddagger}$. In the special case where $\mathcal{O} = \mathcal{O}^{\ddagger}$, we shall call $\mathcal{O}$ a ‡-*order*. In this paper, we characterize ‡-orders over local and global fields—in the local case, we show that either the quaternion algebra is a division algebra or there is a correspondence between maximal or modular lattices and ‡-orders (see Theorems 4.1, 8.1, and 8.2).

The interest in this subject comes from the problem of constructing certain arithmetic sphere packings in $\mathbb{R}^3$. In $\mathbb{R}^2$, one can construct [3] [4] circle packings by considering the action of a Bianchi group $SL(2, \mathcal{O}_K)$ on the real line, where $\mathcal{O}_K$ is the ring of integers of some imaginary quadratic field K. This same method can be used to produce interesting sphere packings in $\mathbb{R}^3$, but requires replacing the Bianchi group $SL(2, \mathcal{O})$ with an appropriate analog $SL^{\ddagger}(2, \mathcal{O})$, where $\mathcal{O}$ is a maximal ‡-order (see [5]).

For this reason, we shall be primarily interested in maximal ‡-orders—that is, ‡-orders not properly contained inside any other ‡-order. Since any order is contained inside a maximal order, it is easy to see that any maximal ‡-order must be of the form $\mathcal{O} \cap \mathcal{O}^{\ddagger}$, where $\mathcal{O}$ is maximal. Therefore, all maximal ‡-orders

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