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On a variant of Pillai's problem II [☆]

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In this paper, we show that there are only finitely many c such that the equation $U_n - V_m = c$ has at least two distinct solutions (n, m), where $\{U_n\}_{n\geqslant 0}$ and $\{V_m\}_{m\geqslant 0}$ are given linear recurrence sequences.

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1. Introduction

A linear recurrence sequence is a sequence $\{U_n\}_{n\geqslant 0}$ such that for some $k \geqslant 1$, we have

$$U_{n+k} = c_1 U_{n+k-1} + \cdots + c_k U_n$$

for all $n \geqslant 0$, where $c_1, \ldots, c_k$ are given complex numbers with $c_k \neq 0$. When $c_1, \ldots, c_k$ are integers and $U_0, \ldots, U_{k-1}$ are also integers, U_n is an integer for all $n \geqslant 0$ and we say that $\{U_n\}_{n\geqslant 0}$ is defined over the integers. In what follows we will always assume that $\{U_n\}_{n\geqslant 0}$ is defined over the integers.

It is known that if we write

$$F(X) = X^k - c_1 X^{k-1} - \cdots - c_k = \prod_{i=1}^{t} (X - \alpha_i)^{\sigma_i},$$

where $\alpha_1, \ldots, \alpha_t$ are distinct complex numbers, and $\sigma_1, \ldots, \sigma_t$ are positive integers whose sum is k, then there exist polynomials $a_1(X), \ldots, a_t(X)$ whose coefficients are in $\mathbb{Q}(\alpha_1, \ldots, \alpha_t)$ such that $a_i(X)$ is of degree at most $\sigma_i - 1$ for $i = 1, \ldots, t$, and such that furthermore the formula

$$U_n = \sum_{i=1}^{t} a_i(n) \alpha_i^n$$

holds for all $n \geqslant 0$. We may certainly assume that $a_i(X)$ is not the zero polynomial for any $i = 1, \ldots, t$. We call $\alpha = \alpha_1$ a dominant root of $\{U_n\}_{n\geqslant 0}$, if $|\alpha_1| > |\alpha_2| \geqslant \ldots \geqslant |\alpha_t|$. In this case the sequence $\{U_n\}_{n\geqslant 0}$ is said to satisfy the dominant root condition.

This paper is a follow-up to our previous work [6], in which we found all integers c admitting at least two distinct representations of the form $F_n - T_m$ for some positive integers $n \geqslant 2$ and $m \geqslant 2$. Here we denote by $\{F_n\}_{n\geqslant 0}$ the sequence of Fibonacci numbers given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geqslant 0$, and denote by $\{T_m\}_{m\geqslant 0}$ the sequence of Tribonacci numbers given by $T_0 = 0$, $T_1 = T_2 = 1$ and $T_{m+3} = T_m + T_{m+1} + T_{m+2}$ for all $m \geqslant 0$. In [6] the main result is the following:

Theorem 1. *The only integers c having at least two representations of the form $F_n - T_m$ come from the set*

$$\mathcal{C} = \{0, 1, -1, -2, -3, 4, -5, 6, 8, -10, 11, -11, -22, -23, -41, -60, -271\}.$$

Furthermore, for each $c \in \mathcal{C}$ all representations of the form $c = F_n - T_m$ with integers $n \geqslant 2$ and $m \geqslant 2$ are obtained.