Journal of Number Theory 183 (2018) 428–441

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Some divisibility properties of binomial coefficients

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A R T I C L E I N F O

Article history:
Received 14 July 2016
Received in revised form 19 August 2017
Accepted 19 August 2017
Available online 20 September 2017
Communicated by the Principal Editors

MSC:
primary 11B65
secondary 05A10

Keywords:
Binomial coefficients
p-adic valuation
Lucas' theorem
Euler's totient theorem
Bernoulli numbers

A B S T R A C T

In this paper, we aim to give full or partial proofs for the following three conjectures of V. J. W. Guo and C. Krattenthaler: (1) Let $a > b$ be positive integers, α, β be any integers and p be a prime satisfying $\gcd(p, a) = 1$. Then there exist infinitely many positive integers n for which $\binom{an+\alpha}{bn+\beta} \equiv r \pmod{p}$ for all integers r; (2) For any odd prime p, there are no positive integers $a > b$ such that $\binom{an}{bn} \equiv 0 \pmod{pn-1}$ for all $n \geq 1$; (3) For any positive integer m, there exist positive integers a and b such that $am > b$ and $\binom{amn}{bn} \equiv 0 \pmod{an-1}$ for all $n \geq 1$.

1. Introduction

Binomial coefficients constitute an important class of numbers that arise naturally in mathematics, namely as coefficients in the expansion of the polynomial $(x + y)^n$. Accordingly, they appear in various mathematical areas. An elementary property of

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http://dx.doi.org/10.1016/j.jnt.2017.08.005

binomial coefficients is that $\binom{n}{m}$ is divisible by a prime p for all $1 < m < n$ if and only if n is a power of p. A much more technical result due to Lucas asserts that

$$\binom{n}{m} \equiv \binom{n_0}{m_0}\binom{n_1}{m_1}\cdots\binom{n_k}{m_k} \pmod{p},$$

in which $n = n_0+n_1p+\cdots+n_kp^k$ and $m = m_0+m_1p+\cdots+m_kp^k$ are the p-adic expansions of the non-negative integers n and m, respectively. We note that $0 \le m_i, n_i < p$, for all $i = 0, \ldots, k$. In 1819, Babbage [1] revealed the following congruences for all odd prime p:

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^2}.$$

In 1862, Wolstenholme [6] strengthened the identity of Babbage by showing that the same congruence holds modulo p^3 for all primes $p \ge 5$. This identity was further generalized by Ljunggren in 1952 to $\binom{np}{mp} \equiv \binom{n}{m} \pmod{p^3}$ and even more to $\binom{np}{mp}/\binom{n}{m} \equiv 1 \pmod{p^q}$ by Jacobsthal for all positive integers $n > m$ and primes $p \ge 5$, in which p^q is any power of p dividing $p^3mn(n-m)$. Arithmetic properties of binomial coefficients are studied extensively in the literature and we may refer the interested reader to [6] for an account of Wolstenholme's theorem. Recently, Guo and Krattenthaler [2] studied a similar problem and proved the following conjecture of Sun [4].

Theorem 1.1. *Let a and b be positive integers. If $bn+1$ divides $\binom{an+bn}{an}$ for all sufficiently large positive integers n, then each prime factor of a divides b. In other words, if a has a prime factor not dividing b, then there are infinitely many positive integers n for which $bn+1$ does not divide $\binom{an+bn}{an}$.*

They also stated several conjectures among which are the followings. We aim to prove or give partial proofs to these conjectures.

In Section 2, we prove Conjecture 1.2 in special cases, see Theorems 2.1 and 2.2.

Conjecture 1.2 *([2, Conjecture 7.2]). For any odd prime p, there are no positive integers $a > b$ such that*

$$\binom{an}{bn} \equiv 0 \pmod{pn-1}$$

for all $n \ge 1$.

In Section 3, using only properties of the p-adic valuation, we give a full proof for Conjecture 7.3 of [2].

Conjecture 1.3 *([2, Conjecture 7.3]). For any positive integer m, there exist positive integers a and b such that $am > b$ and*