ARTICLE IN PRESS YJNTH:5837

Journal of Number Theory ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On the arithmetic of elliptic curves and a homotopy limit problem

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ARTICLE INFO

Article history:
Received 3 October 2016
Received in revised form 18 August 2017
Accepted 19 August 2017
Available online xxxx
Communicated by S.J. Miller

Keywords:
Elliptic curves
Tate–Shafarevich group
Homotopy limit problem
Motivic cohomology
Etale cohomology

ABSTRACT

In this note, I study a comparison map between a motivic and étale cohomology group of an elliptic curve over $\mathbb{Q}$ just outside the range of Voevodsky's isomorphism theorem. I show that the property of an appropriate version of the map being an isomorphism is equivalent to certain arithmetical properties of the elliptic curve.

1. Introduction

The most well known theorem of motivic homotopy theory is Voevodsky's proof of the Beilinson–Lichtenbaum and Bloch–Kato conjectures [13]. In one form ([13], Theorem 6.17), this result states that for a pointed smooth simplicial scheme X, the natural homomorphism

$$\widetilde{H}^p_{Mot}(X, \mathbb{Z}/\ell(q)) \to \widetilde{H}^p_{\acute{e}t}(X, \mathbb{Z}/\ell(q)) \tag{1}$$

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[1] Igor Kriz was supported by NSF grant DMS 1102614 and by a grant from the Simons Foundation, number 403297.

http://dx.doi.org/10.1016/j.jnt.2017.08.008

is an isomorphism for $p \leq q$ and a monomorphism for $p = q + 1$.

The purpose of the present note is to study the map (1) when X is an elliptic curve over $\mathbb{Q}$, $p = 2$, $q = 1$. In this case, we know from Voevodsky's theorem that (1) is a monomorphism.

Theorem 1. *Let $X = E$ be an elliptic curve defined over $\mathbb{Q}$. Then the canonical homomorphism*

$$\widetilde{H}^2_{Mot}(E, \mathbf{Z}_\ell(1)) \to \widetilde{H}^2_{\text{ét}}(E, \mathbf{Z}_\ell(1))$$

where $\mathbf{Z}_\ell(1)$ denotes the homotopy limit of $\mathbb{Z}/\ell^k(1)$ in the category of motives (resp. étale motives) always has an uncountable cokernel.

The situation changes, however, if we work with finite models. For a large enough set S of primes in $\mathbb{Z}$, an elliptic curve E over $\mathbb{Q}$ has a smooth projective model over $\mathbb{Z}[S^{-1}]$, which we will denote by $E[S^{-1}]$.

Theorem 2. *Let $X = E$ be an elliptic curve defined over $\mathbb{Q}$. Then the canonical homomorphism*

$$\lim_{\substack{\rightarrow \\ S}} \widetilde{H}^2_{Mot}(E[S^{-1}], \mathbf{Z}_\ell(1)) \to \lim_{\substack{\rightarrow \\ S}} \widetilde{H}^2_{\text{ét}}(E[S^{-1}], \mathbf{Z}_\ell(1))$$

is an isomorphism if and only if $Ш(E)_{(\ell)}$ is finite and $rank_{\mathbb{Q}}(E) > 0$.

Remark. Both direct limits in the statement of the Theorem are in fact eventually constant.

Here

$$Ш(E) = \bigcap_{\nu} Ker(H^1(\mathbb{Q}, E) \to H^1(\mathbb{Q}_\nu, E_\nu))$$

(where the intersection is taken over all completions of $\mathbb{Q}$) is the *Tate–Shafarevich group*, the finiteness of which (even at one prime) is equivalent to the vanishing of the discrepancy between the rank of the group of rational points of E and its computable estimate (see, for example, [7] for an introduction).

We see easily (as reviewed in the next section) that for $p = 2$, $q = 1$, (1) is never an isomorphism for $X = S^0$. Therefore, it would never be an isomorphism for an elliptic curve if we took unreduced instead of reduced cohomology. It is worthwhile noting that philosophically speaking, by taking reduced cohomology, the weight of the motive in question increases by 1. If it increased by 2, we would be back in the range of Voevodsky's

Please cite this article in press as: I. Kriz, On the arithmetic of elliptic curves and a homotopy limit problem, J. Number Theory (2018), http://dx.doi.org/10.1016/j.jnt.2017.08.008