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Arithmetic Euler top

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ABSTRACT

The theory of differential equations has an arithmetic analogue in which derivatives of functions are replaced by Fermat quotients of numbers. Many classical differential equations (Riccati, Weierstrass, Painlevé, etc.) were previously shown to possess arithmetic analogues. The paper introduces and studies an arithmetic analogue of the Euler differential equations for the rigid body.

1. Introduction

The theory of differential equations has an arithmetic analogue in which derivatives of functions are replaced by Fermat quotients of numbers. This arithmetic analogue was introduced in [3]; for an exposition of part of the resulting theory we refer to [4]. The present paper fits into the theory developed in [3,4]. However our paper is written so as

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to be entirely self-contained and, in particular, it is independent of [3,4]; the few facts we need from [3,4] will be quickly reviewed here.

Many remarkable classical differential equations have arithmetic analogues. Examples of such classical differential equations are: the Riccati equation [7], the Weierstrass equation [3], the Painlevé VI equation [3,5], Schwarzian equations satisfied by modular forms [3], and linear differential equations corresponding to special connections in Riemannian geometry (such as Chern, Levi-Civita, etc.) [6,7,2,8].

The purpose of the present paper is to develop an arithmetic analogue of the Euler differential equations for the rigid body (the Euler top). This is a system of 3 ordinary (non-linear) differential equations in 3 variables which is one of the simplest examples of *algebraically completely integrable* systems [1]. As such its flow on 3-space, referred to in what follows as the *classical Euler flow*, can be viewed as a derivation δ on the polynomial ring $\mathbb{C}[x_1, x_2, x_3]$ given by the expression

$$(a_2 - a_3)x_2x_3\frac{\partial}{\partial x_1} + (a_3 - a_1)x_3x_1\frac{\partial}{\partial x_2} + (a_1 - a_2)x_1x_2\frac{\partial}{\partial x_3}, \tag{1.1}$$

where $a_1, a_2, a_3 \in \mathbb{C}$ are distinct complex numbers. This flow is trivially seen to have 2 independent prime integrals

$$H_1 = \sum_{i=1}^{3} a_i x_i^2, \quad H_2 = \sum_{i=1}^{3} x_i^2, \tag{1.2}$$

in the sense that

$$\delta H_1 = \delta H_2 = 0. \tag{1.3}$$

For generic $c = (c_1, c_2) \in \mathbb{C}^2$, the loci E_c in 3-space, given by

$$H_1 = c_1, \quad H_2 = c_2, \tag{1.4}$$

are affine elliptic curves. We refer to E_c as the *level sets* of H_1, H_2. Then the classical Euler flow δ is “linearized” when restricted to these level sets E_c in the sense that, if one denotes by δ_c the action of δ as Lie derivative on the 1-forms on E_c and if ω_c is the *canonical* invariant 1-form on E_c (to be defined later in the text), then

$$\delta_c \omega_c = 0. \tag{1.5}$$

Here is our arithmetic analogue of the above.

Let A be a complete discrete valuation ring with maximal ideal generated by an odd prime p and perfect residue field $\mathbb{F} = A/pA$. Let $a_1, a_2, a_3 \in A$ be distinct mod p and consider again H_1, H_2 as in (1.2). Recall from [3] that a *p-derivation* on a ring B (in which p is a non-zero divisor) is a map $\delta : B \to B$ such that the map $\phi : B \to B$, $\phi(b) := b^p + p\delta b$, is a ring homomorphism; ϕ is then referred to as the *Frobenius lift*