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On small bases which admit points with two expansions

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A B S T R A C T

Given two positive integers M and k, let $\mathcal{B}_k(M)$ be the set of bases $q > 1$ such that there exists a real number $x \in [0, M/(q-1)]$ having precisely k different q-expansions over the alphabet $\{0, 1, \ldots, M\}$. In this paper we consider $k = 2$ and investigate the smallest base $q_2(M)$ of $\mathcal{B}_2(M)$. We prove that for $M = 2m$ the smallest base is

$$q_2(M) = \frac{m + 1 + \sqrt{m^2 + 2m + 5}}{2},$$

and for $M = 2m - 1$ the smallest base $q_2(M)$ is the largest positive root of

$$x^4 = (m-1)\,x^3 + 2m\,x^2 + m\,x + 1.$$

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Moreover, for $M = 2$ we show that $q_2(2)$ is also the smallest base of $\mathcal{B}_k(2)$ for all $k \geq 3$.

1. Introduction

Fix a positive integer M. For $q \in (1, M+1]$ the sequence $(d_i) = d_1 d_2 \ldots$ with each $d_i \in \{0, 1, \ldots, M\}$ is called a *q-expansion* of x if

$$x = \sum_{i=1}^{\infty} \frac{d_i}{q^i}.$$

Here the *alphabet* $\{0, 1, \ldots, M\}$ will be fixed throughout the paper. Clearly, x has a q-expansion if and only if $x \in I_{q,M} := [0, M/(q-1)]$.

Since the pioneering work of Rényi [19] and Parry [18], representations of real numbers in non-integer bases have been widely studied in the past thirty years. Different from integer base expansions it is well known that almost every $x \in I_{q,M}$ has a continuum of q-expansions (cf. [20,5]). Moreover, for each $k \in \mathbb{N} \cup \{\aleph_0\}$ there exist $q \in (1, M+1]$ and $x \in I_{q,M}$ such that x has precisely k different q-expansions (see, e.g., [9]). For $k = 1$ the unique q-expansions were extensively investigated. For example, Glendinning and Sidorov showed in [11] that for $M = 1$ when the base q is close to $M + 1$ the set of $x \in I_{q,M}$ with a unique q-expansion has positive Hausdorff dimension (for $M > 1$, see e.g., [16]). De Vries and Komornik [7] investigated the topological properties of the unique q-expansions. Recently, Komornik et al. [13] studied the measure theoretical aspects of the unique q-expansions. For more information on the unique q-expansions we refer the readers to [14,8,15], and the surveys [12,20].

Inspired by the papers of Sidorov [21] and Baker [3] we consider the following sets. For $k \in \mathbb{N} \cup \{\aleph_0\}$, let

$$\mathcal{B}_k(M) := \{q \in (1, M+1] : \text{there exists } x \in I_{q,M} \text{ having precisely } k \text{ different } q\text{-expansions}\}.$$

For $M = 1$ Sidorov [21] determined the smallest base $q_2(1) \approx 1.71064$ of $\mathcal{B}_2(1)$, and proved that the set $\mathcal{B}_2(1)$ contains an interval. Later in [4] Baker and Sidorov considered the smallest base of $\mathcal{B}_k(1)$ for $k \geq 3$ and showed that they are all equal to $q_f(1) \approx 1.75488$. Note that the golden ratio $q_G = (1 + \sqrt{5})/2$ is the smallest base of $\mathcal{B}_{\aleph_0}(1)$ (see Lemma 2.2 below). Recently, Baker [3] showed that the second smallest base of $\mathcal{B}_{\aleph_0}(1)$ is $q_{\aleph_0}(1) \approx 1.64541$. Hence, he concluded that for any $q \in (q_G, q_{\aleph_0}(1))$ each point $x \in I_{q,1}$ either has a unique q-expansion or has a continuum of q-expansions. Based on the ideas