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Integral bases and monogenity of pure fields

István Gaál [*,1], László Remete

University of Debrecen, Mathematical Institute, H-4002 Debrecen, Pf. 400, Hungary

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ABSTRACT

Let m be a square-free integer ($m \neq 0, \pm 1$). We show that the structure of the integral bases of the fields $K = \mathbb{Q}(\sqrt[n]{m})$ is periodic in m. For $3 \leq n \leq 9$ we show that the period length is n^2. We explicitly describe the integral bases, and for $n = 3, 4, 5, 6, 8$ we explicitly calculate the index forms of K. This enables us in many cases to characterize the monogenity of these fields. Using the explicit form of the index forms yields a new technic that enables us to derive new results on monogenity and to get several former results as easy consequences. For $n = 4, 6, 8$ we give an almost complete characterization of the monogenity of pure fields.

1. Introduction

Let m be a square-free integer ($m \neq 0, \pm 1$) and $n \geq 2$ a positive integer. There is an extensive literature of pure fields of type $K = \mathbb{Q}(\sqrt[n]{m})$. (Describing the following results on pure fields we use some basic concepts on monogenity and power integral bases that are detailed in Section 2.)

* Corresponding author.
E-mail addresses: gaal.istvan@unideb.hu (I. Gaál), remete.laszlo@science.unideb.hu (L. Remete).

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B.K. Spearman and K.S. Williams [13] gave an explicit formula for the integral basis of pure cubic fields. B.K. Spearman, Y. Qiduan and J. Yoo [12] showed that if i is a cubefree positive integer then there exist infinitely many pure cubic fields with minimal index equal to i. I. Gaál and T. Szabó [8] studied the behaviour of the minimal indices of pure cubic fields in terms of the discriminant. L. El Fadil [4] gave conditions for the existence of power integral bases of pure cubic fields in terms of the index form equation.

T. Funakura [5] studied the integral basis in pure quartic fields. I. Gaál and L. Remete [7] calculated elements of index 1 (with coefficients $< 10^{1000}$) in pure quartic fields $K = \mathbb{Q}(\sqrt[4]{m})$ for $1 < m < 10^7$, $m \equiv 2, 3 \pmod 4$.

S. Ahmad, T. Nakahara and S.M. Husnine [1] showed that if $m \equiv 1 \pmod 4$, $m \not\equiv \pm 1 \pmod 9$ then $\mathbb{Q}(\sqrt[6]{m})$ is not monogenic. On the other hand [2], if $m \equiv 2, 3 \pmod 4$, $m \not\equiv \pm 1 \pmod 9$ then $\mathbb{Q}(\sqrt[6]{m})$ is monogenic.

A. Hameed and T. Nakahara [9] constructed integral bases of pure octic fields $\mathbb{Q}(\sqrt[8]{m})$. They proved [10] that if $m \equiv 1 \pmod 4$ then $\mathbb{Q}(\sqrt[8]{m})$ is not monogenic. On the other hand A. Hameed, T. Nakahara, S.M. Husnine and S. Ahmad [11] proved that if $m \equiv 2, 3 \pmod 4$ then $\mathbb{Q}(\sqrt[8]{m})$ is monogenic.

A. Hameed, T. Nakahara, S.M. Husnine and S. Ahmad [11] showed that if $m \equiv 2, 3 \pmod 4$ then $\mathbb{Q}(\sqrt[2^n]{m})$ is monogenic, this involves the pure quartic and pure octic fields, as well. Moreover, they showed [11] that if all the prime factors of n divide m then $\mathbb{Q}(\sqrt[n]{m})$ is monogenic.

Our purpose is for $3 \leq n \leq 9$ to give a general characterization of the integral basis of $K = \mathbb{Q}(\sqrt[n]{m})$. We prove that the integral bases of $K = \mathbb{Q}(\sqrt[n]{m})$ are periodic in m. For $3 \leq n \leq 9$ the period length is n^2.

The knowledge of the integral bases makes possible also to compute the sporadic results on the monogenity of these fields. Our method applying the explicit form of the index forms yields a new technic that enables us to obtain new results on the monogenity of these fields and to obtain several former results as easy consequences.

In our Theorems 4, 7, 8 we give an almost complete characterization of the monogenity of pure quartic, sextic and octic fields, respectively. The cubic case is well-known and easy, much less is known about the quintic, septic and nonic cases.

2. Basic concepts about the monogenity of number fields

We recall those concepts [6] that we use throughout. Let α be a primitive integral element of the number field K (that is $K = \mathbb{Q}(\alpha)$) of degree n with ring of integers $\mathbb{Z}_K$. The **index** of α is

$$I(\alpha) = (\mathbb{Z}_K^+ : \mathbb{Z}[\alpha]^+) = \sqrt{\left|\frac{D(\alpha)}{D_K}\right|} = \frac{1}{\sqrt{|D_K|}} \prod_{1 \leq i < j \leq n} \left|\alpha^{(i)} - \alpha^{(j)}\right|,$$