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An explicit formula for the local zeta function of a Laurent polynomial

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ABSTRACT

In a recent paper Zúñiga-Galindo and the author begun the study of the local zeta functions for Laurent polynomials. In this work we continue this study by giving a very explicit formula for the local zeta function associated to a Laurent polynomial f over a p-adic field, when f is weakly non-degenerate with respect to the Newton polytope of f at infinity.

1. Introduction

Local zeta functions were introduced in the 60s by Israel Gel'fand and André Weil. Gel'fand studies these functions over $\mathbb{R}$ with the aim of showing the existence of fundamental solutions to certain partial differential equations with constant coefficients.

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Meanwhile Weil studied p-adic local zeta functions in order to generalize some results of Siegel about quadratic forms, e.g. the Siegel–Poisson formula, see [1]. Significant contributions have since been made, specially in the last three decades, see e.g. [1,4,8] and the references therein.

One of the most powerful tools in the characteristic zero wing of the theory is the resolution of singularities. In particular, after the pioneering work of Varchenko [9], Newton polyhedra techniques have been extensively employed to study local zeta functions as well as their connections with oscillatory integrals, see e.g. [1,5] and the references therein for the Archimedean case, and [2,10,12], among others, in the non-Archimedean case, including the positive characteristic case. In [6,7] we began the study of local zeta functions for a Laurent polynomial f over a p-adic field. There we introduce a Newton polytope at infinity Γ_∞ associated to f, together with a non-degeneracy condition in order to show the existence of a meromorphic continuation to the whole complex plane of the local zeta functions for f. We also obtain asymptotic expansions for p-adic oscillatory integrals attached to Laurent polynomials and give bounds for the size of 'tubular neighborhoods' attached to the polynomials. Our work in [6] is closely related with the recent paper [11], where the authors study the local zeta functions for meromorphic functions.

The main tool used in [6] for the meromorphic continuation of the local zeta functions is a variation of toric resolution of singularities. In the classical case of a polynomial function f, one uses the Newton polyhedron of f to construct a conical decomposition of the first orthant of $\mathbb{R}^n$ into simple cones, this decomposition is called a fan. Then one constructs a toric manifold and a map of it into $\mathbb{R}^n$, which together resolve the singularities of almost all the critical points of f. This strategy was carried out successfully in [6] for the case of Laurent polynomials. However, there is a detail that should be pointed out about our construction. In general, the set of generators of the simple fan may contain extra rays coming from the intersection of $\mathbb{R}^n_+$ with the cones in the original fan, and these extra rays could lead to superfluous candidate poles for the local zeta function, see Example 2.2 and the discussion before Example 2.1.

In this paper we justified the aforementioned fact and show that it may happen only in dimensions above 2. Further work in the description of this new set of rays could be of some interest. In the second part of this work we go back to the referred construction of [6], in particular we review and slightly refine the definition of the conical partition subordinated to the Newton polytope of a non-degenerate Laurent polynomial f. Our goal is to give an explicit formula for the local zeta function attached to a character of the group of units of the local ring of K and a non-degenerate Laurent polynomial f, see Section 4. Furthermore, we provide several examples that we hope may shed more light on our approach in [6,7] and here, to the study of this new type of local zeta functions.