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Imaginary quadratic function fields with ideal class group of prime exponent

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For K, an imaginary quadratic extension of a rational function field over a finite field, in which the infinite place ramifies, we give necessary conditions (illustrating for exponent three) for the ideal class group to have odd prime exponent. For exponent two we classify all such extensions, taking this opportunity to complete the list that we previously had.

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1. Introduction

Consider a rational function field of one variable F over its field of constants $\mathbb{F}_q$, where $\mathbb{F}_q$ is a finite field of $q = p^n$ elements, p a prime number, $n \geq 1$. Once we choose a generator x of F over $\mathbb{F}_q$, it is customary to call those places of F coming from irreducible primes of $A = \mathbb{F}_q[x]$ the finite places, and the unique remaining place is called the infinite place and is denoted by $\mathfrak{p}_\infty$ or $\mathfrak{p}_{1/x}$ because it has uniformizer $1/x$.

Artin [1] considered the analogy $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R} \leftrightarrow A = \mathbb{F}_p[x]$, $F = \mathbb{F}_p(x)$, $F_\infty = \mathbb{F}_p((1/x))$, p an odd prime, and called a quadratic extension K of F *real* if $K \subset F_\infty$, i.e., there are two places of K over $\mathfrak{p}_\infty$ (i.e., $\mathfrak{p}_\infty$ splits completely) and *imaginary* if there is only one place $\mathfrak{P}_\infty$ above $\mathfrak{p}_\infty$. MacRae [6] calls K imaginary if further $\mathfrak{p}_\infty$ ramifies in K. In this article, we follow Artin's definition.

If K is not a geometric quadratic extension of $\mathbb{F}_q(x)$, then K is $\mathbb{F}_{q^2}(x)$, which is understood, so we restrict to geometric imaginary quadratic extensions only.

We can consider the divisor class group $C_0(K)$ of divisors of degree 0 of K or the ideal class group $C(\mathcal{O}_K)$ of the integral closure $\mathcal{O}_K$ of $\mathbb{F}_q[x]$ in K. We know these are finite and $|C(\mathcal{O}_K)| = (\deg \mathfrak{P}_\infty)\,|C_0(K)|$. Thus, when we consider ideal class groups of exponent ℓ, ℓ an odd prime, we automatically reduce to the ramified case.

Often, e.g. in Drinfeld's theory, any chosen place is called the infinite place and others are called finite places, but we stick to this classical terminology, which practically means that any place of degree one can be chosen to be the infinite place, given just the abstract rational function field. Hence K is called [8] a *totally imaginary* extension of F if no place of degree one in F splits in K.

To classify relative extensions K/F, one uses isomorphisms keeping F (and hence $\mathbb{F}_q$) constant, but to shorten the list or to use abstract K's one often uses the notion of isomorphism instead. Here, we classify function fields up to isomorphism.

2. Norms of integral elements in imaginary extensions

Let K be an imaginary extension of $\mathbb{F}_q(x)$ with genus g. There is only one place $\mathfrak{P}_\infty$ which lies over the infinity place $\mathfrak{p}_\infty$ of $\mathbb{F}_q(x)$. The following theorem, provides a bound on the degree of the norm of integral elements.

Theorem 2.1. *[8, Theorem 4, p. 220] If K is an imaginary quadratic extension of $\mathbb{F}_q(x)$, then, for any $\alpha \in K \setminus \mathbb{F}_q[x]$, which is integral over $\mathbb{F}_q[x]$,*

$$\deg N(\alpha) \geq 2g + 1,$$

where N is the norm of α.

Corollary 2.2. *If K is an imaginary quadratic extension of $\mathbb{F}_q(x)$ for which the divisor class group has exponent e, then, for any finite place $\mathfrak{p}$ of $\mathbb{F}_q(x)$ which splits in K,*