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The second moment of sums of coefficients of cusp forms

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Let f and g be weight k holomorphic cusp forms and let $S_f(n)$ and $S_g(n)$ denote the sums of their first n Fourier coefficients. Hafner and Ivić [9] proved asymptotics for $\sum_{n\leq X}|S_f(n)|^2$ and proved that the Classical Conjecture, that $S_f(X) \ll X^{\frac{k-1}{2}+\frac{1}{4}+\epsilon}$, holds on average over long intervals.

In this paper, we introduce and obtain meromorphic continuations for the Dirichlet series $D(s, S_f \times S_g) = \sum S_f(n)\overline{S_g(n)} \times n^{-(s+k-1)}$ and $D(s, S_f \times \overline{S_g}) = \sum_n S_f(n)S_g(n)n^{-(s+k-1)}$. We then prove asymptotics for the smoothed second moment sums $\sum S_f(n)\overline{S_g(n)}e^{-n/X}$, giving a smoothed generalization of [9]. We also attain asymptotics for analogous sums of normalized Fourier coefficients. Our methodology extends to a wide variety of weights and levels, and comparison with [4] indicates very general cancellation between the Rankin–Selberg L-function $L(s, f \times g)$ and convolution sums of the coefficients of f and g.

In forthcoming works, the authors apply the results of this paper to prove the Classical Conjecture on $|S_f(n)|^2$ is true

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on short intervals, and to prove sign change results on $\{S_f(n)\}_{n\in\mathbb{N}}$.

1. Introduction and statement of results

Let f be a holomorphic cusp form on a congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ and of positive weight k, where $k \in \mathbb{Z} \cup (\mathbb{Z} + \frac{1}{2})$. Let the Fourier expansion of f at ∞ be given by

$$f(z) = \sum_{n\geq 1} a(n)e(nz),$$

where $e(z) = e^{2\pi i z}$. In this paper, we consider upper bounds for the second moment of the partial sums of the Fourier coefficients,

$$S_f(n) := \sum_{m\leq n} a(m).$$

Bounds on the coefficients $a(n)$ are of great interest and have wide application. The famous Ramanujan–Petersson conjecture, which was proven to hold for integral weight holomorphic cusp forms as a consequence of Deligne's proof of the Weil Conjecture [6], gives us that $a(n) \ll n^{\frac{k-1}{2}+\epsilon}$ and from this one might naively assume $S_f(X) \ll X^{\frac{k-1}{2}+1+\epsilon}$. However, there is significant cancellation in the sum and we expect the far better bound,

$$S_f(X) \ll X^{\frac{k-1}{2}+\frac{1}{4}+\epsilon}, \tag{1.1}$$

which we refer to as the "Classical Conjecture," echoing Hafner and Ivić in their work [9].

Chandrasekharan and Narasimhan, as a consequence of their much broader work on the average order of arithmetical functions [5,4], proved that the Classical Conjecture is true *on average* by showing that

$$\sum_{n\leq X} |S_f(n)|^2 = CX^{k-1+\frac{3}{2}} + B(X), \tag{1.2}$$

where $B(x)$ is an error term,

$$B(X) = \begin{cases} O(X^k \log^2(X)) \\ \Omega\left(X^{k-\frac{1}{4}} \frac{(\log\log\log X)^3}{\log X}\right), \end{cases} \tag{1.3}$$

and C is the constant,