Journal of Number Theory 173 (2017) 371–377

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Generalized Vandermonde determinants and characterization of divisibility sequences

Stefano Barbero

Department of Mathematics, University of Turin, Via Carlo Alberto 10, 10122, Turin, Italy

A R T I C L E I N F O

Article history:
Received 10 September 2016
Received in revised form 27 September 2016
Accepted 27 September 2016
Available online 8 November 2016
Communicated by D. Goss

MSC:
11B37
11B83

Keywords:
Non-degenerate linear recurrence sequences
Impulse sequences
Divisibility sequences
Vandermonde determinants

A B S T R A C T

We present a different proof of the characterization of non-degenerate linear recurrence sequences, which are also divisibility sequences, given by Van der Poorten, Bézivin, and Pethö in their paper "A Full Characterization of Divisibility Sequences" [1]. Our proof is based on an interesting determinant identity, related to impulse sequences, obtained from the evaluation of a generalized Vandermonde determinant. As a consequence of this new proof we can find a more precise form for the resultant sequence presented in [1], in the general case of non-degenerate divisibility sequences having minimal polynomial with multiple roots.

1. Introduction

Finding properties for non-degenerate linear recurrence sequences which are also divisibility sequences, discovering some kind of deeper structure in them, is a very fascinating research field. The most important attempt to establish their behaviour in an elegant

E-mail address: stefano.barbero@unito.it.

http://dx.doi.org/10.1016/j.jnt.2016.09.025

way was presented in the paper of Van der Poorten, Bézivin, and Pethö [1], where they confirmed what Ward conjectured in his paper [6] about the possibility that every linear divisibility sequence should be a divisor of a resultant sequence. In a field $\mathbb{F}$ of characteristic zero, they considered a non-degenerate linear recurrence sequence $(a_n)_{n=0}^{+\infty}$, whose characteristic polynomial has distinct roots. Using the Hadamard quotient theorem and the theory of exponential polynomials they stated that if such a sequence is a divisibility sequence, then there is a resultant sequence $(\bar{a}_n)_{n=0}^{+\infty}$ such that

$$\forall n \geq 0 \quad a_n \mid \bar{a}_n,$$

where $\bar{a}_n$ has the shape

$$\bar{a}_n = n^k \prod_i \left(\frac{\alpha_i^n - \beta_i^n}{\alpha_i - \beta_i} \right). \tag{1}$$

The aim of this paper is to present an elementary proof of this result based on generalized Vandermonde determinants. First of all we point out an interesting identity connecting the non-degenerate impulse sequences with generalized Vandermonde determinants. Then we use it to restate the main result presented in [1]. We give a refinement and a more precise form for the n-th term of the resultant sequence (1), dealing with the more general case of non-degenerate linear recurrence sequences, which are also divisibility sequences, whose minimal polynomial has multiple roots. From now on we work over a field $\mathbb{F}$ of characteristic zero, using the equivalent notations a_n or $a(n)$ for the n-th term of a given sequence over $\mathbb{F}$. We also remember, once and for all, that we consider a linear recurrence sequence as non-degenerate if the ratio of two distinct roots of its minimal polynomial is not a root of unity, and obviously all the roots are different from zero. Moreover we say that a non-degenerate linear recurrence sequence $(a_n)_{n=0}^{+\infty}$ is a *divisibility sequence* if $a_0 = 0$, $a_1 = 1$ and there exists a subring $\mathcal{A}$ of $\mathbb{F}$, of finite type over $\mathbb{Z}$, such that for all n and $d \geq 1$ we have $\frac{a_{nd}}{a_n} \in \mathcal{A}$ (see e.g. [1,4]).

2. Impulse sequences and generalized Vandermonde determinants

We recall the definition of the particular linear recurrence sequences named *impulse sequences*.

Definition 1. Let $P(x) = \prod_{j=1}^{s}(x - \alpha_j)^{m_j}$, with $\alpha_j \in \mathbb{F}$, m_j integers $m_j \geq 1$, and $r = m_1 + \cdots + m_s$. We define the *impulse sequences* of order r as the non-degenerate linear recurrence sequences $\left(X_n^{(k)}\right)_{n=0}^{+\infty}$, $k = 0, \ldots r-1$, with minimal polynomial $P(x)$, such that their initial conditions are $X_h^{(k)} = \delta_{hk}$, $h = 0, \ldots, r-1$ (δ_{hk} is the usual Kronecker delta).

A very important property which relates every linear recurrence sequence with suitable impulse sequences is stated in the following