Journal of Number Theory 173 (2017) 448–477

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

# Statistics for biquadratic covers of the projective line over finite fields

Elisa Lorenzo [a,*], Giulio Meleleo [b,*], Piermarco Milione [c,*], with an appendix by Alina Bucur [d]

[a] *Universiteit Leiden, Mathematisch Instituut, Niels Bohrweg 1, 2333 CA Leiden, The Netherlands*
[b] *Università degli Studi "Roma Tre", Dipartimento di Matematica e Fisica, Largo San Leonardo Murialdo 1, 00146 Roma, Italy*
[c] *Universitat de Barcelona, Departament d'Àlgebra i Geometria, Gran Via de les Corts Catalanes 585, 08005 Barcelona, Spain*
[d] *University of California, San Diego, Department of Mathematics, 9500 Gilman Drive #0112, CA 92093, La Jolla, USA*

ARTICLE INFO

*Article history:*
Received 29 March 2016
Received in revised form 29 September 2016
Accepted 30 September 2016
Available online 14 November 2016
Communicated by D. Goss

*MSC:*
11G20
14H05
11M50

*Keywords:*
Function fields
Biquadratic curves
Biquadratic covers
Number of points over finite fields
Arithmetic statistics

ABSTRACT

We study the distribution of the traces of the Frobenius endomorphisms of genus $g$ curves which are quartic non-cyclic covers of $\mathbb{P}^1_{\mathbb{F}_q}$, as the curve varies in an irreducible component of the moduli space. We show that for $q$ fixed, the limiting distribution of the traces of Frobenius equals the sum of $q+1$ independent random discrete variables. We also show that when both $g$ and $q$ go to infinity, the normalized trace has a standard complex Gaussian distribution. Finally, we extend these computations to the general case of arbitrary covers of $\mathbb{P}^1_{\mathbb{F}_q}$ with Galois group isomorphic to $r$ copies of $\mathbb{Z}/2\mathbb{Z}$. For $r = 1$ we recover the already known results for the family of hyperelliptic curves.



* Corresponding authors.
*E-mail addresses:* e.lorenzo.garcia@math.leidenuniv.nl (E. Lorenzo), meleleo@mat.uniroma3.it (G. Meleleo), piermarcomilione@gmail.com (P. Milione), alina@math.ucsd.edu (A. Bucur).

http://dx.doi.org/10.1016/j.jnt.2016.09.007
0022-314X/

## 1. Introduction

One of the most influential results in class field theory is Chebotarev's density theorem. As is well known, this result is a deep generalization of the Theorem of Dirichlet about the equidistribution of rational primes in arithmetic progression and provides a complete understanding of the distribution of primes in a fixed Galois number field extension with respect to their splitting behavior (for an interesting discussion of the theorem and its original proof, see [LS96]). In the function field case, the analogous statement is the Sato–Tate conjecture for curves, which studies the distribution of the Frobenius endomorphism of the reduction modulo $p$ of a fixed curve, when the prime $p$ varies.

In order to extend this research line in other directions, several mathematicians considered the following new general problem: given a family of genus $g$ curves over $\mathbb{F}_q$ satisfying certain properties, to understand the distribution of the Frobenius endomorphisms of the curves of the family. This is sometimes called the *vertical Sato–Tate conjecture*, since the prime $p$ is fixed and the curve varies in the family. We can study the limiting distribution in two different ways, depending on whether we let the genus $g$ or the cardinality $q$ of the field tend to infinity. It is then interesting to compare both limit results.

When $g$ is fixed and $q$ goes to infinity, the problem can be solved thanks to Deligne's equidistribution theorem (cf. [KS99]), once one computes the monodromy group of the family, while for the complementary case different techniques are applied depending on the particular family considered. The fluctuation in the number of points at the $g$-limit has been studied for different families of curves, such as:

- hyperelliptic curves, cf. [KR09,BDFL09],
- cyclic trigonal curves, cf. [BDFL09,Xio10],
- general trigonal curves, cf. [Woo12],
- $p$-fold cover of the projective line, [BDFL11],
- $\ell$-covers of the projective line, cf. [BDFL09,BDF+16].

In the present paper, we study the distribution of the number of points over $\mathbb{F}_q$ for a genus $g$ curve $C$ defined over $\mathbb{F}_q$ which is a quartic non-cyclic cover of the projective line $\mathbb{P}^1_{\mathbb{F}_q}$, both at the $q$-limit (for a genus $g$ fixed) and at the $g$-limit (with $q$ fixed). This is the first time that a family of non-cyclic abelian covers has been studied. The distribution obtained is different to the product of probabilities for the family of hyperelliptic curves, which at first sight could be guessed. Therefore, the study of this family seems to be the first natural step in order to understand the general abelian case.

Let $\mathcal{B}_g(\mathbb{F}_q)$ be the family of genus $g$ quartic non-cyclic cover of the projective line $\mathbb{P}^1_{\mathbb{F}_q}$, and consider the following decomposition

$$\mathcal{B}_g(\mathbb{F}_q) = \bigcup_{g_1+g_2+g_3=g} \mathcal{B}_{(g_1,g_2,g_3)}(\mathbb{F}_q)$$