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Distinct distances on regular varieties over finite fields

Do Duy Hieu Pham Van Thang

Abstract

In this paper we study some generalized versions of a recent result due to Covert, Koh, and Pi (2015). More precisely, we prove that if a subset $\mathcal{E}$ in a *regular* variety satisfies $|\mathcal{E}| \gg q^{\frac{d-1}{2}+\frac{1}{k-1}}$, then

$$\Delta_{k,F}(\mathcal{E}) := \left\{ F(\mathbf{x}^1 + \cdots + \mathbf{x}^k) \colon \mathbf{x}^i \in \mathcal{E}, 1 \le i \le k \right\} \supseteq \mathbb{F}_q \setminus \{0\},$$

for some certain families of polynomials $F(\mathbf{x}) \in \mathbb{F}_q[x_1, \ldots, x_d]$.

1 Introduction

Let $\mathbb{F}_q$ be a finite field of order q, where q is a prime power. Let $D(\mathbf{x}) = x_1^2 + \cdots + x_d^2$ be a polynomial in $\mathbb{F}_q[x_1, \ldots, x_d]$. For $\mathcal{E} \subset \mathbb{F}_q^d$, we define the D-distance set of $\mathcal{E}$ to be

$$\Delta(\mathcal{E}) = \{D(\mathbf{x} - \mathbf{y}) \colon \mathbf{x}, \mathbf{y} \in \mathcal{E}\}.$$

There are various papers studying the cardinality of $\Delta(\mathcal{E})$, see for example [3, 9, 5, 4, 10, 12] and references therein. In this paper, we are interested in the case when $\mathcal{E}$ is a subset in a *regular* variety. Let us first start with a definition of regular varieties which is taken from [4]

Definition 1.1. *For $\mathcal{E} \subseteq \mathbb{F}_q^d$, let $\mathbf{1}_\mathcal{E}$ denote the characteristic function on $\mathcal{E}$. Let $F(\mathbf{x}) \in \mathbb{F}_q[x_1, \ldots, x_d]$ be a polynomial. The variety $\mathcal{V} := \{\mathbf{x} \in \mathbb{F}_q^d \colon F(\mathbf{x}) = 0\}$ is called a regular variety if $|\mathcal{V}| \asymp q^{d-1}$ and $\widehat{\mathbf{1}_\mathcal{V}(\mathbf{m})} \ll q^{-(d+1)/2}$ for all $\mathbf{m} \in \mathbb{F}_q^d \setminus \mathbf{0}$, where*

$$\widehat{\mathbf{1}_\mathcal{V}(\mathbf{m})} = \frac{1}{q^d} \sum_{\mathbf{x} \in \mathbb{F}_q^d} \chi(-\mathbf{m} \cdot \mathbf{x}) \mathbf{1}_\mathcal{V}(\mathbf{x}).$$

Here and throughout, $X \asymp Y$ means that there exist positive absolute constants C_1 and C_2 which do not depend on X, Y, and q such that $C_1 Y < X < C_2 Y$, $X \ll Y$ means that there exists a positive absolute constant C that does not depend on X, Y and q such that $X \le CY$, and $X = o(Y)$ means that $X/Y \to 0$ as $q \to \infty$, where X, Y are viewed as functions in q.

There are several examples of regular varieties as follows: