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Supercongruences arising from basic hypergeometric series

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ABSTRACT

We employ some formulas on basic hypergeometric series and hypergeometric series and the p-adic Gamma function to establish several new supercongruences.

1. Introduction

Supercongruences are congruences which happen to hold modulo some higher power of a prime p. The topic of supercongruences is related to the p-adic Gamma function, elliptic curves, Gauss and Jacobi sums, hypergeometric series, modular forms, Calabi–Yau manifolds, many special polynomials and some sophisticated combinatorial identities

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involving harmonic numbers (see, for example, [13]). Many supercongruences were conjectured by a lot of mathematicians including van Hamme [18], Zudilin [19], Chan et al. [1], Z.-W. Sun [16,17] and Z.-H. Sun [14,15]. In particular, using a sequence of orthogonal polynomials, van Hamme [18] set up the following congruence

$$\sum_{k=0}^{\frac{p-1}{2}}(4k+1)\binom{-1/2}{k}^4 \equiv p \pmod{p^3} \tag{1.1}$$

for each $p \geq 3$. Here and below, we use the notation $A \equiv B \pmod{p^l}$ if $(A-B)/p^l$ is a p-integer for $A, B \in \mathbb{Q}$. We shall give a new proof of (1.1) in the sequel.

In [8], applying some hypergeometric evaluation identities, Long obtained several supercongruences related to special valuations of truncated hypergeometric series. For example, she employed a strange evaluation of Gosper to get the following supercongruences conjectured by van Hamme [18, (J.2)]: for any prime $p > 3$, we have

$$\sum_{k=0}^{\frac{p-1}{2}}(-1)^k\frac{6k+1}{4^k}\binom{-1/2}{k}^3 \equiv -\frac{p}{\Gamma_p(\frac{1}{2})^2} \pmod{p^3}$$

where $\Gamma_p(x)$ is the p-adic Gamma function (see Section 2). In 2014, using p-adic Gamma function and formulas on hypergeometric series, Long and Ramakrishna [9] established many supercongruences. In particular, they proved that for any prime $p \geq 5$, there hold

$$\sum_{k=0}^{p-1}(6k+1)\binom{-1/3}{k}^6 \equiv \begin{cases} -p\Gamma_p(\frac{1}{3})^9 & \text{if } p \equiv 1 \pmod 6 \\ -\frac{10}{27}p^4\Gamma_p(\frac{1}{3})^9 & \text{if } p \equiv 5 \pmod 6 \end{cases} \pmod{p^6}$$

and

$${}_3F_2\left(\begin{matrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ & 1 & 1\end{matrix};1\right)_{p-1} \equiv \begin{cases} \Gamma_p(\frac{1}{3})^6 \pmod{p^3} & \text{if } p \equiv 1 \pmod 6 \\ -\frac{p^2}{3}\Gamma_p(\frac{1}{3})^6 \pmod{p^3} & \text{if } p \equiv 5 \pmod 6 \end{cases},$$

where ${}_{r+1}F_s(\cdot)_n$ is the truncated hypergeometric series given by

$${}_{r+1}F_s\left(\begin{matrix}a_0 & a_1 & \ldots & a_r \\ b_1 & b_2 & \ldots & b_s\end{matrix};z\right)_n := \sum_{k=0}^{n}\frac{(a_0)_k(a_1)_k\cdots(a_r)_k}{k!(b_1)_k\cdots(b_s)_k}z^k$$

and $(z)_n$ is defined by

$$(z)_0 = 1,\ (z)_n = z(z+1)\cdots(z+n-1) \text{ for } n \geq 1.$$

Some other supercongruences on truncated hypergeometric series were obtained by different authors (see, for example, [4–6,10–12]).

Motivated by the work of Long [8] and Long and Ramakrishna [9], we shall establish the following supercongruence which appears to be new.

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