Journal of Pure and Applied Algebra ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Stability results for projective modules over Rees algebras

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A R T I C L E I N F O

Article history:
Received 1 May 2017
Received in revised form 5 February 2018
Available online xxxx
Communicated by V. Suresh

MSC:
Primary: 13C10; 19A13; secondary: 13A30

A B S T R A C T

We provide a class of commutative Noetherian domains R of dimension d such that every finitely generated projective R-module P of rank d splits off a free summand of rank one. On this class, we also show that P is cancellative. At the end we give some applications to the number of generators of a module over the Rees algebras.

1. Introduction

Let R be a commutative Noetherian ring of Krull dimension d. A classical result of Serre [20] says that every finitely generated projective R-module P of rank $> d$ splits off a free summand. This is the best possible result in general as it is evidenced from the well-known example of "the tangent bundle over real algebraic sphere of dimension two". Therefore the question "splitting off a free summand" becomes subtle when $rank(P) = d$. If R is a reduced affine algebra over an algebraically closed field, then for a rank d projective R-module P, M.P. Murthy [13] defined an obstruction class $c_d(P)$ in the group $F^dK_0(A)$. Further assuming $F^dK_0(A)$ has no $(d-1)!$ torsion, he proved that $c_d(P) = 0$ if and only if P splits off a free summand of rank one.

For a commutative Noetherian ring R of dimension d, Bhatwadekar–Raja Sridharan ([3], [4]) defined an obstruction group called Euler class group, denoted by $\mathrm{E}^d(R)$. Assume $\mathbb{Q} \subset R$, then given a projective R-module of rank d, Bhatwadekar–Raja Sridharan defined an obstruction class $e_d(P)$ and proved $e_d(P) = 0$ in $\mathrm{E}^d(R)$ if and only if P splits off a free summand of rank one. Later, for a smooth scheme X of dimension n, Barge–Morel [1] defined the Chow–Witt group $\widetilde{CH}^j(X)$ $(j \geq 0)$ and associated to each vector bundle E of rank n with trivial determinant an Euler class $\widetilde{c}_n(E)$ in $\widetilde{CH}^n(X)$. Let A be a smooth affine domain of

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https://doi.org/10.1016/j.jpaa.2018.02.031
0022-4049/

dimension n and P a finitely generated projective A-module of rank n. Then it was proved that $\widetilde{c_n}(P) = 0$ if and only if $P \tilde{\to} Q \oplus A$ for $n = 2$ in [1] (see also [8]), $n = 3$ in [7] and $n \geq 4$ in [12].

A recent result of Marco Schlichting [19] proved a similar kind of result for a commutative Noetherian ring R of dimension d all of whose residue fields are infinite. Precisely, given a rank d oriented projective R-module P, he defined a class $e(P)$ in $H^d_{Zar}(R, \mathcal{K}^{MW}_d)$ such that $e(P) = 0$ if and only if P splits off a free summand of rank one.

One of the aims of this article is to provide a class of examples of commutative Noetherian rings R of dimension d such that every rank d projective R-module splits off a free summand of rank one. We prove the following:

Theorem 1.1. *Let R be a commutative Noetherian domain of dimension $d-1$ $(d \geq 1)$ and I an ideal of R. Define $A := R[It]$ or $R[It, t^{-1}]$ (note that $dim(A) \leq d$). Let P be a projective A-module of rank d. Then $P \tilde{\to} Q \oplus A$ for some projective A-module Q.*

In particular, if $\mathbb{Q} \subset A$, then the obstruction class $e_d(P)$ defined by Bhatwadekar–Raja Sridharan [4] is zero in $\mathrm{E}^d(A)$. Also, in the view of Schlichting's result [19], if we assume that all residue fields of A are infinite, then $e(P)$ defined by Schlichting is zero in $H^d_{Zar}(A, \mathcal{K}^{MW}_d)$. For $A = R[t]$ and for birational overrings of $R[t]$, a similar type of result is proved by Plumstead [14] and Rao ([16], [17]) respectively.

In this direction, a parallel problem is "the cancellation problem". Let P be a projective module over a commutative Noetherian ring R of dimension d such that $rank(P) > d$. Then Bass [2] proved that P is cancellative i.e. $P \oplus Q \tilde{\to} P' \oplus Q \Rightarrow P \tilde{\to} P'$. Again this is the best possible result in general as it is evidenced by the same well-known example "tangent bundle over the real algebraic sphere of dimension two". However Suslin ([22]) proved that if R is an affine algebra of dimension d over an algebraically closed field, then every projective R-module of rank d is cancellative. We enlarge the class of rings by proving the following result.

Theorem 1.2. *Let R be a commutative Noetherian domain of dimension $d-1$ $(d \geq 1)$ and I an ideal of R. Define $A := R[It]$ or $R[It, t^{-1}]$. Then every finitely generated projective A-module of rank d is cancellative.*

For $A = R[t]$ and for birational overrings of $R[t]$, a similar type of result is proved by Plumstead [14] and Rao ([16], [17]) respectively.

The following result follows from our result Theorem 1.2 and a result of Wiemers [24, Theorem].

Corollary 1.3. *Let R be a commutative Noetherian domain of dimension $d-1$ $(d \geq 1)$ such that $1/d! \in R$ and I an ideal of R. Define $A := R[It]$ or $R[It, t^{-1}]$. Then every finitely generated projective $A[X_1, \ldots, X_n]$-module of rank d is cancellative.*

As an application of our result, we prove the following result.

Theorem 1.4. *Let R be a commutative Noetherian domain of dimension d and I an ideal of R. Let M be a finitely generated module over $A := R[It]$ or $R[It, t^{-1}]$. Then M is generated by $e(M) := Sup_{\mathfrak{p}}\{\mu_{\mathfrak{p}}(M) + dim(A/\mathfrak{p})\}$ elements.*

Let A be a domain of dimension n, $R = A[X_1, \ldots, X_m]$ and I the ideal of R generated by $(X_1, \ldots, X_m)$. Then $R[It] = A[X_1, \ldots, X_m, X_1t, \ldots, X_mt]$. Note that in this case $R[It]$ becomes a monoid algebra $A[M]$, where M is the monoid generated by $(X_1, \ldots, X_m, X_1t, \ldots, X_mt)$. In this case Gubeladze [10] conjectured that every projective $A[M]$-module of rank $> n$ splits off a free summand of rank one. In Theorem 1.1, we have verified it affirmatively but our rank-dimension condition is not optimal. We note that the second