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On Brauer p-dimensions and absolute Brauer p-dimensions of Henselian fields

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A R T I C L E I N F O

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A B S T R A C T

This paper determines the Brauer p-dimension $\mathrm{Brd}_p(K)$ and the absolute Brauer p-dimension $\mathrm{abrd}_p(K)$ of a Henselian valued field (K, v), for a prime $p \neq \mathrm{char}(\widehat{K})$, under restrictions on the residue field $\widehat{K}$, such as the condition $\mathrm{abrd}_p(\widehat{K}) = 0$. It describes the set Σ_0 of sequences $\mathrm{abrd}_p(E), \mathrm{Brd}_p(E)$, $p \in \mathbb{P}$, where $\mathbb{P}$ is the set of prime numbers and E runs across the class of Henselian fields with $\mathrm{char}(\widehat{E}) = 0$ and a projective absolute Galois group $\mathcal{G}_{\widehat{E}}$. Specifically, Σ_0 contains a sequence $a_p, b_p \in \mathbb{N} \cup \{0, \infty\}$, $p \in \mathbb{P}$, whenever $a_2 \leq 2b_2$ and $a_p \geq b_p$, for each p. Similar results are obtained in characteristic $q > 0$.

1. Introduction

Let E be a field, $s(E)$ the class of finite-dimensional associative central simple E-algebras, $d(E)$ the subclass of division algebras $D \in s(E)$, and for each $A \in s(E)$, let $[A]$ be the equivalence class of A in the Brauer group $\mathrm{Br}(E)$. By Wedderburn's Structure Theorem (cf. [32], Sect. 3.5), $[A]$ has a representative $D_A \in d(E)$ which is uniquely determined by A, up-to an E-isomorphism; this implies the dimension $[A\colon E]$ is a square of a positive integer $\deg(A)$, the degree of A. Also, it is known that $\mathrm{Br}(E)$ is an abelian torsion group, so it decomposes into the direct sum of its p-components $\mathrm{Br}(E)_p$, taken over the set $\mathbb{P}$ of prime numbers (see [32], Sect. 14.4). The Schur index $\mathrm{ind}(D) = \deg(D_A)$ and the exponent $\exp(A)$, i.e. the order of $[A]$ in $\mathrm{Br}(E)$ (called also a period of A), are important invariants of both D_A and $[A]$. Their general relations and behaviour under scalar extensions of finite degrees are described as follows (cf. [32], Sects. 13.4, 14.4 and 15.2):

(1.1) (a) $\exp(A) \mid \mathrm{ind}(A)$ and $p \mid \exp(A)$, for each $p \in \mathbb{P}$ dividing $\mathrm{ind}(A)$. For any $B \in s(E)$ with g.c.d$\{\mathrm{ind}(B), \mathrm{ind}(A)\} = 1$, $\mathrm{ind}(A \otimes_E B) = \mathrm{ind}(A).\mathrm{ind}(B)$; when A, $B \in d(E)$, the tensor product $A \otimes_E B$ lies in $d(E)$;

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(b) ind(A) and ind($A \otimes_E R$) divide ind($A \otimes_E R$)$[R\colon E]$ and ind(A), respectively, for each finite field extension R/E of degree $[R\colon E]$.

As shown by Brauer (see, e.g., [32], Sect. 19.6), there exists a field F, such that $d(F)$ contains an algebra $D_{m,n}$ with $\exp(D_{m,n}) = m$ and $\mathrm{ind}(D_{m,n}) = n$, whenever (n, m) is a Brauer pair, i.e. $n, m \in \mathbb{N}$, $m \mid n$, and $p' \mid m$ in case $p' \in \mathbb{P}$ and $p' \mid n$. It is known, however, that index-exponent relations over a number of frequently used fields are subject to much tougher restrictions than those described by (1.1) (a). The Brauer p-dimensions $\mathrm{Brd}_p(E)$, $p \in \mathbb{P}$, of a field E and their supremum $\mathrm{Brd}(E)$, the Brauer dimension of E, contain essential information about the Brauer pairs $(\mathrm{ind}(A), \exp(A))$, $A \in s(E)$. We say that $\mathrm{Brd}_p(E)$ is finite and equal to n, if n is the least integer ≥ 0 satisfying the divisibility condition $\mathrm{ind}(D) \mid \exp(D)^n$, for each $D \in d(E)$ with $[D] \in \mathrm{Br}(E)_p$. When no such n exists, we put $\mathrm{Brd}_p(E) = \infty$. It follows from (1.1) (a) that $\mathrm{Brd}(E) \leq 1$ if and only if $\mathrm{ind}(D) = \exp(D)$, for each $D \in d(E)$. We have $\mathrm{Brd}_p(E) = 0$, for a given $p \in \mathbb{P}$, if and only if $\mathrm{Br}(E)_p = \{0\}$; in particular, $\mathrm{Brd}(E) = 0 \leftrightarrow \mathrm{Br}(E) = \{0\}$.

By an absolute Brauer p-dimension of E, we mean the supremum $\mathrm{abrd}_p(E) = \sup\{\mathrm{Brd}_p(R)\colon R \in \mathrm{Fe}(E)\}$, where $\mathrm{Fe}(E)$ is the set of finite extensions of E in its separable closure E_{sep}. The absolute Brauer dimension of E is defined by $\mathrm{abrd}(E) = \sup\{\mathrm{Brd}(R)\colon R \in \mathrm{Fe}(E)\}$. When $\mathrm{abrd}_p(E) = 0$, the p-cohomological dimension $\mathrm{cd}_p(\mathcal{G}_E)$ of the absolute Galois group $\mathcal{G}_E = \mathcal{G}(E_{\mathrm{sep}}/E)$ is ≤ 1, and the converse holds if E is perfect or $p \neq \mathrm{char}(E)$ (see [16], Theorem 6.1.8, or [35], Ch. II, 3.1). We have $\mathrm{Brd}_p(E) = \mathrm{abrd}_p(E) = 1$, $p \in \mathbb{P}$, if E is a global or local field (see [33], (31.4) and (32.19)), or the function field of an algebraic surface over an algebraically closed field E_0 [18], [24]. Then $\mathrm{Br}(E)_p$, $p \in \mathbb{P}$, possess nonzero divisible subgroups (see [33], (31.8) and (32.13), [28], (16.1), and [32], Sect. 15.1, Corollary a), so (n, n), $n \in \mathbb{N}$, are all index-exponent E-pairs. When E_1 is the function field of an algebraic curve over a perfect pseudo algebraically closed (PAC) field E_0, $\mathrm{Brd}_p(E_1) = \mathrm{abrd}_p(E_1) = \mathrm{cd}_p(\mathcal{G}_{E_0})$, $p \in \mathbb{P}$ [11]. Note also that $\mathrm{abrd}_p(F_k) < p^{k-1}$, $p \in \mathbb{P}$, if F_k is a field of C_k-type, for some $k \in \mathbb{N}$ [27].

This paper studies the values of sequences $\mathrm{abrd}_p(E), \mathrm{Brd}_p(E)$, $p \in \mathbb{P}$, of fields E. It presents a research motivated by problems concerning index-exponent relations and Brauer p-dimensions of finitely-generated field extensions. One of these problems, posed in [2], Sect. 4, can be stated as follows:

(1.2) Prove whether the class of fields of finite Brauer dimensions is closed under the formation of finitely-generated field extensions.

2. Statements of the main results

The interest in the p-dimensions $\mathrm{abrd}_p(E)$, $\mathrm{Brd}_p(E)$, $p \in \mathbb{P}$, of a field E is due to the fact that $\mathrm{abrd}_p(E)$ is a lower bound of $\mathrm{Brd}_p(F)$, for any $p \in \mathbb{P}$ and every finitely-generated purely transcendental extension F/E (see [6], Theorem 2.1). Our first main result, stated below, describes the set of sequences $\mathrm{abrd}_p(E), \mathrm{Brd}_p(E)$, $p \in \mathbb{P}$, defined over the class of fields E of zero characteristic, for which $\mathrm{Brd}_2(E) = \infty$ or $\mathrm{abrd}_2(E) \leq 2\mathrm{Brd}_2(E) < \infty$ (this generalizes [5], Theorem 2.3). It does the same in characteristic $q > 0$, for a large class of fields containing finitely many roots of unity:

Theorem 2.1. *Let* $(\bar{a}, \bar{b}) = a_p, b_p \in \mathbb{N}_\infty\colon\ p \in \mathbb{P}$, *be a sequence with* $a_p \geq b_p$, *for each* p, *where* $\mathbb{N}_\infty = \mathbb{N} \cup \{0, \infty\}$. *Let also* $a_2 \leq 2b_2$ *or* $b_2 = \infty$. *Then:*

(a) *There exists a field* ∇_0, *such that* $\mathrm{char}(\nabla_0) = 0$ *and* $(\mathrm{abrd}_p(\nabla_0), \mathrm{Brd}_p(\nabla_0)) = (a_p, b_p)$, *for every* $p \in \mathbb{P}$*;*

(b) *There is a field* ∇_q *with* $\mathrm{char}(\nabla_q) = q > 0$ *and* $(\mathrm{abrd}_p(\nabla_q), \mathrm{Brd}_p(\nabla_q)) = (a_p, b_p)$, $p \in \mathbb{P}$, *provided that* $b_q \leq a_q \leq b_q + 1$ *if* $b_q < \infty$, *that* $a_p = 0$ *whenever* $b_p = 0$, *and* $a_p \leq 2b_p$ *whenever* $p \mid (q-1)$ *and* $b_p < \infty$.

It seems unknown whether there is a field E containing a primitive p-th root of unity, and such that $\mathrm{abrd}_p(E) > 1 + 2\mathrm{Brd}_p(E)$, for some $p \in \mathbb{N}$. Therefore, it is worth noting that the fields ∇_0 and ∇_q whose existence is obtained by our proof of Theorem 2.1 have also the following properties (see page 18):

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