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A degree bound for families of rational curves on surfaces

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ABSTRACT

We give an upper bound for the degree of rational curves in a family that covers a given birationally ruled surface in projective space. The upper bound is stated in terms of the degree, sectional genus and arithmetic genus of the surface. We introduce an algorithm for constructing examples where the upper bound is tight. As an application of our methods we improve an inequality on lattice polygons.

1. Introduction

A *parametrization* of a rational surface $S \subset \mathbb{P}^n$ is a birational map

$$f : \mathbb{C}^2 \dashrightarrow S \subset \mathbb{P}^n, \quad (s,t) \mapsto (f_0(s,t) : \ldots : f_n(s,t)).$$

The *parametric degree* of S is defined as the minimum of the set of integers of the form $\max\{\deg f_i | 0 \leq i \leq n\}$ for some birational map $f : \mathbb{C}^2 \dashrightarrow S$.

An upper bound for the parametric degree over an algebraically closed field of characteristic 0 is given in [10, Theorem 9] in terms of the sectional genus and degree of S. In [11, Theorem 20] bounds for the parametric degree over perfect fields are expressed in terms of the level and keel (Definition 1). The upper bound in [10, Theorem 9] can be interpreted as an upper bound on the level. The analysis of [11] applied to toric surfaces led to new inequalities for invariants of lattice polygons in [5]. In [3, Section 2.7] it is conjectured that the inequality can be improved by taking into account the number of vertices. In [6, Theorem 5] these inequalities for lattice polygons are translated to inequalities of rational surfaces and in [6, Section 4] the conjecture of [3] is restated in the context of rational surfaces.

In this paper we generalize the level and keel for rational surfaces in [11, Section 3] to birationally ruled surfaces (this generalization is also posed as an open question in [6, Section 1]). Instead of the parametric degree we now consider the minimal family degree (defined in §4). Theorem 1 gives an upper bound for the

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level of a birationally ruled surface $S \subset \mathbb{P}^n$ in terms of the sectional genus, degree and arithmetic genus. As Corollary 1 we obtain an upper bound for the minimal family degree. If S is rational, then our upper bound for the level coincides with the upper bound for the level in [10, Lemma 8]. However, in order to generalize this bound we give an alternative proof. This proof enables us to make a case distinction on the invariants of S, which improves the upper bound for the level. Moreover, these methods enables us to prove the correctness of Algorithm 1 that outputs examples where our upper bound is attained. Thus we show that our upper bound for the level is tight in a combinatorial sense. This algorithm is simple but has a non-trivial correctness proof. These methods generalize the inequality [6, Theorem 5] to birationally ruled surfaces. If we restrict our generalized inequality to toric surfaces, we obtain an improved inequality involving lattice polygons as conjectured in [3, Section 2.7], [6, Section 4]. In light of the historical context, one might ask whether this inequality can be improved using the language of lattice geometry.

I would like to end this introduction with some additional remarks on the degree of minimal parametrizations. Let $s(f) := \max\{\deg_s f_i | 0 \le i \le n\}$, $t(f) := \max\{\deg_t f_i | 0 \le i \le n\}$ and we assume without loss of generality that $t(f) \ge s(f)$. The *parametric bi-degree* of S is defined as the minimum of $(s(f), t(f))$ among all birational maps $f : \mathbb{C}^2 \dashrightarrow S$ with respect to the lexicographic order on ordered pairs of integers. If $S \subset \mathbb{P}^n$ attains at least two minimal families, then the parametric bi-degree of S equals $(v(S), v(S))$ where $v(S)$ is minimal family degree [7, Theorem 17]. Thus in this case our upper bound for the minimal family degree translates into an upper bound of the parametric bi-degree. If S carries only one minimal family, then an upper bound for the parametric bi-degree is still open. In this case we also have to incorporate the keel in addition to the sectional genus, degree and arithmetic genus of S.

2. Intersection theory

We recall some intersection theory and this section can be omitted by the expert.

The *Neron–Severi group* $N(X)$ of a non-singular projective surface X can be defined as the group of divisors modulo numerical equivalence. This group admits a bilinear intersection product

$$\cdot : N(X) \times N(X) \longrightarrow \mathbb{Z}.$$

The *Picard number* of X is defined as the rank of $N(X)$. The *Neron–Severi theorem* states that the Picard number is finite. For proofs in the next section we implicitly also consider $N(X) \otimes \mathbb{R}$. Moreover, we switch between the linear and numerical equivalence class of a divisor where needed.

The class of an *exceptional curve* E in $N(X)$ is characterized by

$$E^2 = E \cdot K = -1,$$

where K is the canonical divisor class of X. *Castelnuovo's contractibility criterion* states that for all exceptional curves E there exists a contraction map

$$f : X \longrightarrow Y,$$

such that $f(E) = p$ with p a smooth point and $(X \setminus E) \longrightarrow (Y \setminus p)$ is an isomorphism via f. The assignment of Neron–Severi groups is functorial such that

$$f^* : N(Y) \longrightarrow N(X).$$

The groups are related by

$$N(X) \cong N(Y) \oplus \mathbb{Z}\langle E\rangle,$$