Journal of Pure and Applied Algebra 223 (2019) 349–368

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

# Invariant polynomials on truncated multicurrent algebras

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ARTICLE INFO

*Article history:*
Received 7 June 2017
Received in revised form 20 March 2018
Available online 29 March 2018
Communicated by D. Nakano

*MSC:*
Primary: 17B05; 17B08; secondary: 17B35; 17B70

ABSTRACT

We construct invariant polynomials on truncated multicurrent algebras, which are Lie algebras of the form $\mathfrak{g} \otimes_{\mathbb{F}} \mathbb{F}[t_1, \dots, t_\ell]/I$, where $\mathfrak{g}$ is a finite-dimensional Lie algebra over a field $\mathbb{F}$ of characteristic zero, and $I$ is a finite-codimensional ideal of $\mathbb{F}[t_1, \dots, t_\ell]$ generated by monomials. In particular, when $\mathfrak{g}$ is semisimple and $\mathbb{F}$ is algebraically closed, we construct a set of algebraically independent generators for the algebra of invariant polynomials. In addition, we describe a transversal slice to the space of regular orbits in $\mathfrak{g} \otimes_{\mathbb{F}} \mathbb{F}[t_1, \dots, t_\ell]/I$. As an application of our main result, we show that the center of the universal enveloping algebra of $\mathfrak{g} \otimes_{\mathbb{F}} \mathbb{F}[t_1, \dots, t_\ell]/I$ acts trivially on all irreducible finite-dimensional representations provided $I$ has codimension at least two.



## 1. Introduction

### 1.1. Motivation

Suppose $A$ is a finitely-generated commutative associative unital algebra over a field $\mathbb{F}$ of characteristic zero. All tensor products will be over $\mathbb{F}$. Furthermore, suppose that $\Gamma$ is a finite group acting on $A$ and on a finite-dimensional Lie algebra $\mathfrak{g}$ by automorphisms. The corresponding *equivariant map algebra* $(\mathfrak{g} \otimes A)^\Gamma$ can be viewed as the Lie algebra of $\Gamma$-equivariant algebraic maps from $\operatorname{Spec} A$ to $\mathfrak{g}$. Equivariant map algebras are a large class of Lie algebras generalizing loop algebras and current algebras, which are vital to the theory of affine Lie algebras, and are an extremely active area of research. We refer the reader to the survey [15] for an overview of the field.

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[1] The first author was supported by CNPq grant 232462/2014-3.
[2] The second author was supported by Discovery Grant RGPIN-2017-03854 from the Natural Sciences and Engineering Research Council of Canada.

https://doi.org/10.1016/j.jpaa.2018.03.014
0022-4049/

In the case that $\Gamma$ is abelian and acts freely on $\operatorname{maxSpec} A$, $\mathbb{F}$ is algebraically closed, and $\mathfrak{g}$ is semisimple, one can use the twisting and untwisting functors defined in [8] to reduce the study of finite-dimensional representations of $(\mathfrak{g} \otimes A)^\Gamma$ to the study of representations of the corresponding *untwisted* map algebra $\mathfrak{g} \otimes A$, where $\Gamma$ is trivial (see [8, Th. 2.10]).

Now assume that $\Gamma$ is trivial and $\mathfrak{g}$ is semisimple. Since $A$ is finitely generated, we may assume that $A = \mathbb{F}[t_1, \dots, t_\ell]/I$ for some ideal $I$ of $\mathbb{F}[t_1, \dots, t_\ell]$. In particular, maximal ideals of $A$ correspond to maximal ideals of $\mathbb{F}[t_1, \dots, t_\ell]$ containing $I$. It is known that any finite-dimensional $\mathfrak{g} \otimes A$-module is a tensor product of modules supported at single points, that is, modules annihilated by $\mathfrak{g} \otimes (\mathsf{m}^N + I)$ for some maximal ideal $\mathsf{m}$ of $A$ containing $I$, and some $N \in \mathbb{N}$. (This follows, for example, from [8, Prop. 2.4].) Suppose $V$ is a $\mathfrak{g} \otimes A$-module annihilated by $\mathfrak{g} \otimes (\mathsf{m}^N + I)$. Translating if necessary, we may assume that $\mathsf{m} = (t_1, \dots, t_\ell)$ is the maximal ideal corresponding to the origin. Since $\mathsf{m}^N + I$ annihilates $V$, so does $\mathsf{m}^N$. Hence, $V$ is naturally a module for the quotient $\mathfrak{g} \otimes \mathbb{F}[t_1, \dots, t_\ell]/\mathsf{m}^N$. It follows that we can reduce the study of finite-dimensional $\mathfrak{g} \otimes A$-modules to the case $A = \mathbb{F}[t_1, \dots, t_\ell]/I$, where $I$ is an ideal of $\mathbb{F}[t_1, \dots, t_\ell]$ of finite codimension, generated by monomials. We call these Lie algebras *truncated multicurrent algebras*.

In the case $\ell = 1$, the Lie algebras $\mathfrak{g} \otimes \mathbb{F}[t]/(t^{n+1})$, called *truncated current algebras*, or *generalized Takiff algebras*, have appeared in many places in the literature. Most relevant to the current paper is the work of Takiff, who considered invariant polynomials in the case $n = 1$ in [19]. Raïs and Tauvel considered the case of arbitrary $n$ in [17], as did Geoffriau in [9,10]. More recently, graded modules for Takiff algebras were investigated in [4], highest-weight theory for truncated current algebras were considered by Wilson in [20], and connections to the geometric Langlands program were studied by Kamgarpour in [12].

Motivated by the above discussion, in the current paper we study the structure of truncated multicurrent algebras. A particularly useful tool in the representation theory of Lie algebras has been the action of the center of the universal enveloping algebra. On the other hand, the Duflo isomorphism is an algebra isomorphism between the center of the universal enveloping algebra of a finite-dimensional Lie algebra and the invariants in its symmetric algebra (see [7]). The goal of the current paper is to describe this space of invariants for truncated multicurrent algebras.

### 1.2. Main results

Suppose $\mathbb{F}$ is a field of characteristic zero, $\mathfrak{g}$ is a finite-dimensional Lie algebra, and $\ell$ is a positive integer. Define a partial order on the set $\Omega = \mathbb{N}^\ell$ by

$$(n_1, \dots, n_\ell) \le (m_1, \dots, m_\ell) \iff n_i \le m_i \quad \text{for all } i \in \{1, \dots, \ell\}.$$

Suppose $\Omega_0$ is a subset of $\Omega$ that is invariant under the action of $\Omega$ on itself by componentwise addition, and such that $\Omega_1 = \Omega \setminus \Omega_0$ is finite. Then the span $\mathbb{F}\langle t^\omega \mid \omega \in \Omega_0\rangle$ is an ideal of $\mathbb{F}[t_1, \dots, t_\ell]$ and we define

$$A = \mathbb{F}[t_1, \dots, t_\ell]/\mathbb{F}\langle t^\omega \mid \omega \in \Omega_0\rangle.$$

In Section 3.1, we associate to any polynomial $p \in S(\mathfrak{g})$ a family $p_\omega$, $\omega \in \Omega$, of elements of $S(\mathfrak{g} \otimes A)$. Suppose that $p \in S(\mathfrak{g})^\mathfrak{g}$, that is, $p$ is invariant under the action of $\mathfrak{g}$ on $S(\mathfrak{g})$ induced by the adjoint action. We then show, in Proposition 3.8, that $p_\omega \in S(\mathfrak{g} \otimes A)^{\mathfrak{g} \otimes A}$ for certain values of $\omega$. In particular, if $\Omega_1$ has a greatest element $\mu$, which we will assume for the remainder of this introduction, then, for all $k \in \mathbb{N}$,

$$p \in S^k(\mathfrak{g})^\mathfrak{g} \implies p_\omega \in S^k(\mathfrak{g} \otimes A)^{\mathfrak{g} \otimes A} \text{ for } \omega \in k\mu - \Omega_1.$$

(Note that $\Omega_1$ has a greatest element if and only if $A$ is a symmetric algebra. If the greatest element is $\mu$, the corresponding trace map is projection onto $A_\mu$.) In other words, given invariant polynomials in $S(\mathfrak{g})$,