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# Local rings whose modules are almost serial ☆

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A B S T R A C T

The present paper is a sequel to our previous work on almost uniserial rings and modules, which appeared in the Journal of Algebra in 2016; it studies rings over which every (left and right) module is almost serial. A module is almost uniserial if any two of its submodules are either comparable in inclusion or isomorphic. And a module is almost serial if it is a direct sum of almost uniserial modules. The results of the paper are inspired by a characterization of Artinian serial rings as rings having all left (or right) modules serial. We prove that if $R$ is a local ring and all left $R$-modules are almost serial then $R$ is an Artinian ring which is uniserial either on the left or on the right. We also produce a connection between local rings having all left and right modules almost serial, local balanced rings studied by Dlab and Ringel and local Köthe rings. Finally we prove Morita invariance of the almost serial property and list some consequences.
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## 1. Introduction

In this paper, all rings have identity elements and all modules are unitary modules. An *Artinian* (resp., *Noetherian*) ring is a ring which is both left and right Artinian (resp., Noetherian). A *principal ideal ring* is a ring which is both left and right principal ideal ring. Also, a ring whose lattice of left ideals is linearly ordered under inclusion, is called a *left uniserial ring*. A *uniserial ring* is a ring which is both left and right uniserial. Note that commutative uniserial rings are also known as *valuation rings*. A left $R$-module $M$ is called *uniserial* if its submodules are linearly ordered by inclusion. Also a left $R$-module $M$ is called *serial* if it is a direct sum of uniserial modules. Thus a left (resp., right) uniserial ring is a ring which is uniserial

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as a left (resp., right) module. A *left* (resp., *right*) *serial ring* is a ring which is serial as a left (resp., right) module. Also a ring $R$ is called *serial* if it is both a left and a right serial ring.

Rings, over which all modules are serial, were first systematically consider by G. Köthe and T. Nakayama (for the main properties of serial rings and modules, we refer to [25]). Köthe in [16] showed that the modules over an Artinian principal ideal rings (which are a special case of serial rings) are direct sums of cyclic submodules. Furthermore, if a commutative Artinian ring has the property that all its modules are direct sums of cyclic modules, then it is necessarily a principal ideal ring. Later Cohen and Kaplansky [7] determined that, for a commutative ring $R$, all $R$-modules are direct sums of cyclic submodules if and only if $R$ is an Artinian principal ideal ring. The corresponding problem in the non-commutative case is still open (see [21, Appendix B, Problem 2.48] and [14, page 40, Problem 1]). Asano [2,3] proved that a commutative ring $R$ is Artinian serial if and only if $R$ is an Artinian principal ideal ring. A *left* (resp., *right*) *Köthe ring* is a ring $R$ such that every left (resp., right) $R$-module decomposes as a direct sum of cyclic submodules. According to Chase [6, Theorem 4.4], every ring enjoying the property that *"left modules are direct sums of finitely generated modules"* is left Artinian and every left module possesses an indecomposable decomposition. The converse was established by Warfield [24, Theorem 3] for commutative rings, and Zimmermann-Huisgen [26, Corollary 2] generalized Warfield's result for arbitrary rings. Thus any left Köthe ring is left Artinian. But a left Artinian principal left ideal ring $R$ need not be a left Köthe ring, even if $R$ is a local ring (see Faith [12, page 212, Remark (2)]). On the other hand, Nakayama [17, page 289] gave an example of a right Köthe ring $R$ which is not a principal right ideal ring.

Also, Nakayama in [18, Theorem 17] proved that if $R$ is an Artinian serial ring, then every left (right) $R$-module is a serial module. The converse is obtained by Skornyakov in [23]. Thus we have the following theorem:

**Theorem 1.1.** (Nakayama–Skornyakov) *A ring $R$ is Artinian serial if and only if every left (right) $R$-modules are serial.*

Most recently the notion of "almost uniserial rings and modules" has been introduced and studied in Behboodi–Roointan [5] as a straightforward common generalization of left uniserial rings and left principal ideal domains. We say that an $R$-module $M$ is *almost uniserial* if any two non-isomorphic submodules are linearly ordered by inclusion, i.e., for every pair $N$, $N'$ of submodules of $M$ either $N \subseteq N'$, or $N' \subseteq N$, or $N \cong N'$. Also, an $R$-module $M$ is called *almost serial* if it is a direct sum of almost uniserial modules. A ring $R$ is *almost left uniserial* (resp., *almost left serial*) if ${}_RR$ is an almost uniserial (resp., almost serial) module. Every uniserial module is almost uniserial, but the converse is not true in general. For instance, the $\mathbb{Z}$-module $\mathbb{Z}$ is almost uniserial, but it is not a uniserial $\mathbb{Z}$-module.

The above Nakayama–Skornyakov Theorem motivates us to study the following questions:

**Question 1.** *Which rings $R$ have the property that every left $R$-module is almost serial?*

**Question 2.** *Which rings $R$ have the property that every left and right $R$-module is almost serial?*

Throughout this paper $(R, \mathcal{M})$ will be a local ring with maximal ideal $\mathcal{M}$. For a left (resp., right) $R$-module $M$ we denote by $E({}_RM)$ (resp., $E(M_R)$) the injective hull of $M$. If the module $M$ is of finite length, then we denote by length$(M)$ the length of $M$.

The goal of this paper is to discuss the above questions in the case $R$ is a local ring. In Section 2, we find a necessary condition for the local ring $R$ to have the property that every left $R$-module is almost serial (such rings must be right uniserial or left uniserial) (see Theorem 2.8). This yields that the following are equivalent for a commutative ring $R$: (1) Every $R$-module is almost serial; (2) Every $R$-module is serial; (3) $R$ is an Artinian serial ring (see Corollary 2.9). Also, we present the following main theorem of Section 2.

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