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The Hilbert series and a-invariant of circle invariants

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Abstract

Let V be a finite-dimensional representation of the complex circle $\mathbb{C}^\times$ determined by a weight vector $\boldsymbol{a} \in \mathbb{Z}^n$. We study the Hilbert series $\mathrm{Hilb}_{\boldsymbol{a}}(t)$ of the graded algebra $\mathbb{C}[V]^{\mathbb{C}^\times_{\boldsymbol{a}}}$ of polynomial $\mathbb{C}^\times$-invariants in terms of the weight vector $\boldsymbol{a}$ of the $\mathbb{C}^\times$-action. In particular, we give explicit formulas for $\mathrm{Hilb}_{\boldsymbol{a}}(t)$ as well as the first four coefficients of the Laurent expansion of $\mathrm{Hilb}_{\boldsymbol{a}}(t)$ at $t = 1$. The naive formulas for these coefficients have removable singularities when weights pairwise coincide. Identifying these cancelations, the Laurent coefficients are expressed using partial Schur polynomials that are independently symmetric in two sets of variables. We similarly give an explicit formula for the a-invariant of $\mathbb{C}[V]^{\mathbb{C}^\times_{\boldsymbol{a}}}$ in the case that this algebra is Gorenstein. As an application, we give methods to identify weight vectors with Gorenstein and non-Gorenstein invariant algebras.

Keywords: Hilbert series, circle invariants, a-invariant, Gorenstein ring, Schur polynomial
2010 MSC: Primary 13A50; Secondary 13H10, 05E05.

1. Introduction

Let V be a finite-dimensional representation of a complex reductive group G and let $R = \mathbb{C}[V]^G$ denote the algebra of G-invariant polynomials. It is well known that this algebra is a finitely generated graded algebra $R = \oplus_{m=0}^{\infty} R_m$ such that $R_0 = \mathbb{C}$. The *Hilbert series* of R is the generating function

$$\mathrm{Hilb}_R(t) = \sum_{m=0}^{\infty} \dim_{\mathbb{C}}(R_m)\, t^m.$$

The Hilbert series is known to be rational with a pole at $t = 1$ of order $\dim(R)$, the Krull dimension of R. It therefore admits a Laurent expansion of the form

$$\mathrm{Hilb}_R(t) = \sum_{m=0}^{\infty} \gamma_m(R)(1-t)^{m-\dim R}, \tag{1.1}$$

see [8, Proposition 1.4.5 and Lemma 1.4.6]. The *a-invariant* $a(R)$ of R is defined to be the degree of $\mathrm{Hilb}_R(t)$, i.e. the degree of the numerator minus the degree of the denominator.

The Hilbert series $\mathrm{Hilb}_R(t)$ contains important information about the algebra R and is a relatively accessible quantity. It is used in constructive invariant theory, e.g., when computing generators and relations for R (see for example [8, Section 2.6] and [21, Chapter 2]). Similarly, the coefficients $\gamma_m(R)$ are often of significance. For instance when G is a finite group, $\gamma_0(R) = 1/|G|$ and $\gamma_1(R)$ determines the number of pseudoreflections in G; see [26, Lemma 2.4.4] or [5,

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