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Parabolic subgroup orbits on finite root systems

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ABSTRACT

Oshima's Lemma describes the orbits of parabolic subgroups of irreducible finite Weyl groups on crystallographic root systems. This note generalises that result to all root systems of finite Coxeter groups, and provides a self contained proof, independent of the representation theory of semisimple complex Lie algebras.

0. Introduction

This note concerns a lemma (Lemma 1.2 below) of Oshima on orbits of standard parabolic subgroups of an irreducible finite Weyl group acting on its crystallographic root system. That result is proved in [10, Lemma 4.3] using results from [9] about the representation theory of semisimple complex Lie algebras. It is applied in [10], and plays a crucial role in [4], in the study of conjugacy classes of reflection subgroups of finite Weyl groups.

The purpose of this note is to give several reformulations and generalisations of the lemma which apply to general root systems of finite Coxeter groups and to provide an elementary self-contained proof of the generalised lemma. Though the statements are uniform, the proof involves some casewise analysis.

This note is organised as follows. In Section 1, we briefly state Oshima's lemma (Lemma 1.2) and our generalisation (Proposition 1.4). Section 2 gives more details on the notions involved in the formulation of these results and their proofs. Section 3 reduces the proofs of both results to showing that Proposition 1.4(a) holds for at least one root system of each irreducible finite Coxeter group and proves Proposition 1.4(a) for dihedral groups. Section 4 proves Proposition 1.4(a) for crystallographic root systems of finite Weyl groups, using elementary properties of root strings. By the classification of irreducible finite Coxeter groups, the proof of Proposition 1.4(a) in general is reduced to the cases of W of type H_3 and H_4; these are treated in

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Section 5 by using foldings of Coxeter graphs to reduce to types D_6 and E_8, where the result is known from Section 4.

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1. Orbits of parabolic subgroups on finite root systems

1.1. Oshima's lemma

Let $\Phi \subseteq V$ be a root system of a finite Coxeter system (W, S), and let Π be a set of simple roots corresponding to S (see Section 2 for details on terminology). For $\alpha \in \mathbb{R}\Phi$ and $\beta \in \Pi$, define the *root coefficient* $\alpha[\beta] \in \mathbb{R}$ by $\alpha = \sum_{\beta\in\Pi} \alpha[\beta]\beta$; define the *support* of α to be $\operatorname{supp}(\alpha) := \{\, \beta \in \Pi \mid \alpha[\beta] \neq 0 \,\}$.

The following result is proved in [10, Lemma 4.3] using facts from [9] about the representation theory of semisimple complex Lie algebras.

Lemma 1.2. *Assume above that W is an irreducible finite Weyl group and Φ is crystallographic. Fix $\Delta \subseteq \Pi$, scalars $c_\beta \in \mathbb{R}$ for $\beta \in \Delta$, not all zero, and $l \in \mathbb{R}$. Let*

$$X := \{\, \gamma \in \Phi \mid \langle \gamma, \gamma \rangle^{1/2} = l \text{ and } \gamma[\beta] = c_\beta \text{ for all } \beta \in \Delta \,\}.$$

If X is non-empty, then it is a single $W_{\Pi\setminus\Delta}$-orbit of roots where $W_{\Pi\setminus\Delta}$ is the standard parabolic subgroup with simple roots $\Pi \setminus \Delta$. Equivalently, $|X \cap \mathscr{C}_{\Pi\setminus\Delta}| \leq 1$ where $\mathscr{C}_{\Pi\setminus\Delta}$ is the closed fundamental chamber of $W_{\Pi\setminus\Delta}$.

1.3. The result below gives reformulations of Lemma 1.2 which are valid for all (possibly non-crystallographic or reducible) finite Coxeter groups.

Proposition 1.4. *Let Φ be a root system of a finite Coxeter group W, with simple roots Π and corresponding simple reflections S. For $J \subseteq S$, let W_J denote the standard parabolic subgroup of W generated by J, and $\mathscr{C}_{W_J}$ denote its closed fundamental chamber.*

(a) *Let $J \subseteq S$. If $\alpha \in \Phi \setminus \Phi_J$, then $W\alpha \cap (\alpha + \mathbb{R}\Pi_J) = W_J\alpha$.*
(b) *Let $J \subseteq S$. If $\alpha, \beta \in (\Phi \setminus \Phi_J) \cap \mathscr{C}_{W_J}$ and $\beta \in W\alpha \cap (\alpha + \mathbb{R}\Pi_J)$, then $\alpha = \beta$.*
(c) *Let Φ' be the root system of a parabolic subgroup W' of W. Then for any $\beta \in \Phi \setminus \Phi'$, one has $W\beta \cap (\beta + \mathbb{R}\Phi') = W'\beta$.*

Remarks 1.5. (1) The assumption in (a) that $\alpha \in \Phi \setminus \Phi_J$ cannot be weakened to $\alpha \in \Phi$. For example, let Φ be of type A_3 in $\mathbb{R}^4$ with simple roots $\alpha_i := e_i - e_{i+1}$ for $i = 1, \ldots, 3$, $\Pi_J = \{\alpha_1, \alpha_3\}$ and $\alpha = \alpha_1$. Then $\alpha_3 \in \Phi \cap (\alpha_1 + \mathbb{R}\Pi_J)$ but $\alpha_3 \notin W_J\alpha_1 = \{\pm\alpha_1\}$.

(2) The proposition does not extend as stated to infinite Coxeter groups. For example, take (W, S) to be irreducible affine of type $\widetilde{A}_3$ with $\Pi = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$ where α_i and α_j are joined in the Coxeter graph if $i - j \in \{\pm 1\} \pmod 4$. Let $\delta := \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$ denote the standard indivisible isotropic root (which is not in Φ; see [8]). Take $\Pi_J = \{\, \alpha_1, \alpha_3 \,\}$. Let $\alpha = 2\delta + \alpha_1 \in \Phi \setminus \Phi_J$. Then $2\delta + \alpha_3 \in W\alpha \cap (\alpha + \mathbb{R}\Pi_J)$ but $2\delta + \alpha_3 \notin W_J\alpha$ by (1), since W fixes δ.

(3) It may also be shown that if $v \in \mathscr{C}_W$, $w \in W$ and $J \subseteq S$, then $Wv \cap (v + \mathbb{R}\Pi_J) = W_J v$. This statement generalises to possibly infinite Coxeter groups; see [6, Lemma 2.4(d)].

(4) We do not know any natural result which generalises both Proposition 1.4(a) and (3). Also, there is no known extension of Proposition 1.4(a) or (3) to unitary reflection groups.

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