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Asymptotic behaviors in the homology of symmetric group and finite general linear group quandles

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ABSTRACT

A quandle is an algebraic structure which attempts to generalize group conjugation. These structures have been studied extensively due to their connections with knot theory, algebraic combinatorics, and other fields. In this work, we approach the study of quandles from the perspective of the representation theory of categories. Namely, we look at collections of conjugacy classes of the symmetric groups and the finite general linear groups, and prove that they carry the structure of FI-quandles (resp. VIC(q)-quandles). As applications, we prove statements about the homology of these quandles, and construct FI-module and VIC(q)-module invariants of links.

1. Introduction

A **quandle** is a set X paired with a binary operation $\triangleright$ satisfying the following:

1. $x \triangleright x = x$ for all $x \in X$;
2. $y \mapsto y \triangleright x$ is a bijection for all $x \in X$;
3. $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$ for all $x, y, z \in X$.

For instance, if G is any group, then G becomes a quandle with operation $x \triangleright y = yxy^{-1}$. While group conjugation may be the most obvious, and perhaps the most motivating, example of a quandle, these objects have been shown to appear all throughout algebra and topology. For instance, one can find applications of quandles to knot theory [22,14,8–10], algebraic geometry [38], and algebraic combinatorics [12]. In [9], a theory of quandle homology was introduced, building off previous work of Fenn, Rourke, and Sanderson [16]. Since then, there has been a large amount of interest directed towards proving facts about these homology groups. The purpose of this paper is to study quandles and their homology from a new perspective: that of asymptotic algebra.

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Let FI denote the category whose objects are sets of the form $[n] = \{1, \ldots, n\}$ and whose morphisms are injections. In their seminal work [5], Church, Ellenberg, and Farb introduced the notion of an FI-module. It was shown that these modules have a plethora of applications to topology, arithmetic, and algebraic combinatorics. An FI-module over a commutative ring k is a functor from the FI to the category of k-modules. Put another way, an FI-module is an object (in an abelian category) which encodes an infinite family of $\mathfrak{S}_n$-representations, where n is allowed to vary. Finite generation of an FI-module is then shown to imply remarkably strong facts about the symmetric group representations which constitute it (see Definition 2.12 and Theorem 2.18 or [5,7], for example). These results stress the following philosophy, which we will use in this work: Given a family of algebraic objects which display some kind of asymptotically regular behavior, there is a single object, finitely generated in some abelian category, which encodes the entire family.

To begin to state the results of this work, we start with the symmetric group. Recall that conjugacy classes of the symmetric group $\mathfrak{S}_n$ are in natural bijection with partitions of n (see Definition 3.1). Let λ be a partition of m which does not have any 1's, called **primitive** in the present work (see Definition 3.1), and let c_λ be the corresponding conjugacy class. Then for each $n \geq m$, we can define c_λ^n as the conjugacy class of $\mathfrak{S}_n$ obtained from c_λ by adding $n - m$ 1-cycles. We begin with the following.

Theorem A. *Let λ be a primitive partition of a fixed integer m. Then the assignment*

$$n \mapsto c_\lambda^n$$

can be extended to a functor from the FI *to the category of quandles.*

As an application of this theorem, we will be able to prove asymptotic facts about the homology of the quandles c_λ^n (see Definition 2.5).

Theorem B. *Let λ be a primitive partition of a fixed integer m, let k be a commutative Noetherian ring, and let $i \geq 0$ be an integer. Then the assignment*

$$n \mapsto H_i^Q(c_\lambda^n; k)$$

can be extended to a finitely generated FI*-module (see Definition 2.12). In particular,*

1. *If k is a field, then there exists a polynomial $p_{Q,i}(T) \in \mathbb{Q}[T]$ of degree $\leq (i \cdot m)$ such that for all $n \gg 0$,*

$$p_{Q,i}(n) = \dim_k H_i^Q(c_\lambda^n; k)$$

2. *For each n, let $\mathfrak{a}_{Q,i,n} \subseteq k$ be the ideal generated by non-zero-divisors which annihilate $H_i^Q(c_\lambda^n; k)$. Then for $n \gg 0$, $\mathfrak{a}_{Q,i,n}$ is independent of n. In particular, if $k = \mathbb{Z}$, then there exists an integer $e_{Q,i}$, independent of n, such that $e_{Q,i}$ is the exponent of the abelian group $H_i^Q(c_\lambda^n)$, for $n \gg 0$.*

As a second application, we construct FI-module invariants of links. In [22], Joyce associates to each oriented link L a quandle $\mathcal{K}(L)$ known as the **fundamental quandle** of L (see Example 2.4). One is then motivated to construct invariants of the link L by looking at the Hom-sets, $\mathrm{Hom}(\mathcal{K}(L), X)$, where X is quandle. These so-called **quandle colorings** of L have been studied extensively [8,13,14]. For instance, it can be shown that the Alexander polynomial of links can be recovered from examining certain quandles [14]. We will prove the following.

Theorem C. *Let L be an oriented link, and let λ be a primitive partition of some fixed integer m. Then there exists a finitely generated* FI*-module $V^{L,\lambda}$ over $\mathbb{Z}$ satisfying,*