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Braces and symmetric groups with special conditions

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A B S T R A C T

We study left braces satisfying special conditions, or identities. We are particularly interested in the impact of conditions like **Raut** and **lri** on the properties of the left brace and its associated solution of the Yang–Baxter equation (YBE). We show that the solution (G, r_G) of the YBE associated to the structure group $G = G(X, r)$ (with the natural structure of a left brace) of a nontrivial solution (X, r) of the YBE has multipermutation level 2 if and only if G satisfies **lri**. It is known that every (left) brace with **lri** satisfies condition **Raut**. We prove that for a graded Jacobson radical ring with no elements of additive order two the conditions **lri** and **Raut** are equivalent. We construct a finite two-sided brace with condition **Raut** which does not satisfy **lri**. We show that a finitely generated two-sided brace which satisfies **lri** has a finite multipermutation level which is bounded by the number of its generators.

1. Introduction

Braces where introduced by Rump [10] as a generalization of Jacobson radical rings to study involutive non-degenerate set-theoretic solutions of the Yang–Baxter equation, an important equation in Mathematical Physics that lies in the foundation of quantum groups [8]. A close relation between braces and symmetric groups (involutive braided groups, [13]) was shown by the second author in [6].

Drinfeld [3] suggested to study set-theoretic solutions of the Yang–Baxter equation, that is a pair (X, r), where X is a nonempty set and $r\colon X \times X \longrightarrow X \times X$ is a bijective map such that the braid relation

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$$r_1 \circ r_2 \circ r_1 = r_2 \circ r_1 \circ r_2,$$

holds in X^3, where $r_1 = r \times \mathrm{id}_X : X^3 \longrightarrow X^3$ and $r_2 = \mathrm{id}_X \times r : X^3 \longrightarrow X^3$.

A solution (X, r) is involutive if $r^2 = \mathrm{id}_{X^2}$. We shall write

$$r(a, b) = ({}^a b, a^b) \quad \text{for } a, b \in X.$$

Consider the maps $\mathcal{L}_x, \mathcal{R}_x : X \to X$ defined by

$$\mathcal{L}_x(y) = {}^x y \quad \text{and} \quad \mathcal{R}_x(y) = y^x,$$

for all $x, y \in X$. A solution (X, r) is *non-degenerate* if the maps $\mathcal{L}_x$ and $\mathcal{R}_x$ are bijective for all $x \in X$.

Convention. In this paper "*a solution*" means "an involutive non-degenerate set-theoretic solution of the Yang–Baxter equation".

Recall that *a left brace* is a set B joint with two binary operations, a sum $+$ and a multiplication $\cdot$, such that $(B, +)$ is an abelian group, $(B, \cdot)$ is a group and

$$a \cdot (b + c) + a = a \cdot b + a \cdot c, \tag{1.1}$$

for all $a, b, c \in B$. *A right brace* is defined similarly, but replacing the equality (1.1) by $(b+c) \cdot a + a = b \cdot a + c \cdot a$. If $(B, +, \cdot)$ is both a left and a right brace (for the same operations), then it is called *a two-sided brace.*

It is known that if B is a left brace, and 0 and 1, respectively, denote the neutral elements with respect to the two operations "+" and "·" in B, then $0 = 1$.

In any left brace B one defines the operation $*$ by the rule:

$$a * b = a \cdot b - a - b, \; a, b \in B. \tag{1.2}$$

Remark 1.1. It is easy to check that $*$ is left distributive with respect to the sum $+$. Moreover, $(B, +, \cdot)$ is a two-sided brace if and only if

$$(a + b) * c = a * c + b * c, \quad \forall a, b, c \in B. \tag{1.3}$$

In general, the operation $*$ is not right distributive, nor associative, but it satisfies the following condition

$$(a * b + a + b) * c = a * (b * c) + a * c + b * c, \quad \forall a, b, c \in B, \tag{1.4}$$

see the original definition of right brace of Rump [10, Definition 2]. Moreover, $(B, +, \cdot)$ is a two-sided brace if and only if $(B, +, *)$ is a Jacobson radical ring, [10].

Every left brace B has *a canonically associated solution of the YBE*, denoted by (B, r_B), where

$$\begin{array}{rccc} r_B \colon & B \times B & \longrightarrow & B \times B \\ & (a, b) & \mapsto & ({}^a b, a^b) \end{array},$$

with ${}^a b := ab - a$ and $a^b := (ab - a)^{-1} ab$, for all $a, b \in B$. Moreover, the map

$$\begin{array}{rccc} \mathcal{L} \colon & (B, \cdot) & \longrightarrow & \mathrm{Aut}(B, +) \\ & a & \mapsto & \mathcal{L}_a \colon B \to B \\ & & & b \mapsto {}^a b \end{array}$$

is a homomorphism of groups (see [2]).