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Graded Steinberg algebras and partial actions

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ABSTRACT

Given a graded ample Hausdorff groupoid, we realise its graded Steinberg algebra as a partial skew inverse semigroup ring. We use this to show that for a partial action of a discrete group on a locally compact Hausdorff topological space which is totally disconnected, the Steinberg algebra of the associated groupoid is graded isomorphic to the corresponding partial skew group ring. We show that there is a one-to-one correspondence between the open invariant subsets of the topological space and the graded ideals of the partial skew group ring. We also consider the algebraic version of the partial C^*-algebra of an abelian group and realise it as a partial skew group ring via a partial action of the group on a topological space. Applications to the theory of Leavitt path algebras are given.

1. Introduction

The notion of crossed product by a partial action has its origin in the concept of crossed product by a partial automorphism introduced by Exel in [17]. Crossed products of C^*-algebras by partial actions of discrete groups were defined in [26] by McClanahan. Skew group rings were introduced by Dokuchaev and Exel in [16] as algebraic analogues of C^*-crossed products by partial actions. The latter algebras are a powerful tool in the study of operator algebras (see [17–19,28]), and so it is important to realise C^*-algebras as partial crossed products (see [8,21] for example), as one can then benefit from the established theory about partial crossed products.

Sieben [30] introduced the notion of a crossed product by an action of an inverse semigroup on a C^*-algebra using covariant representations. Later, a definition of crossed product for actions of inverse semigroups on C^*-algebras, without resorting to covariant representations was presented in [22]. The algebraic version for actions of inverse semigroups on algebras were investigated [10].

Recently Steinberg algebras were introduced in [13,31] as an algebraisation of the groupoid C^*-algebras first studied by Renault [29]. Steinberg algebras include Leavitt and Kumjian–Pask algebras as well as inverse

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semigroup algebras (see [14,15,31]). These classes of algebras have been attracting significant attention, with particular interest in the graded ideal structures of these algebras.

In this note we relate these two class of algebras. Starting from a graded ample Hausdorff groupoid $\mathcal{G}$ and an open invariant subset $U \subseteq \mathcal{G}^{(0)}$, we establish a graded isomorphism

$$A_R(\mathcal{G}_U) \cong_{\mathrm{gr}} C_R(U) \rtimes \mathcal{G}^{(h)}. \tag{1.1}$$

Here $\mathcal{G}^{(h)}$ is the inverse semigroup of graded compact open bisections of $\mathcal{G}$ which acts partially on U, $C_R(U) \rtimes \mathcal{G}^{(h)}$ is the corresponding partial skew inverse semigroup ring and $A_R(\mathcal{G}_U)$ is the Steinberg algebra associated to the groupoid $\mathcal{G}_U = r^{-1}(U)$. In particular, we have a graded isomorphism $A_R(\mathcal{G}) \cong_{\mathrm{gr}} C_R(\mathcal{G}^{(0)}) \rtimes_\pi \mathcal{G}^{(h)}$ (see Theorem 2.3).

Let X be a locally compact Hausdorff topological space which is totally disconnected and $\phi = (\phi_g, X_g, X)_{g \in G}$ a partial action of a group G on X such that each X_g is clopen subset of X. Then we have an induced partial action of G on $C_R(X)$. Denote by $\mathcal{G}_X = \bigcup_{g \in G} g \times X_g$ the G-graded groupoid given in (3.2) associated to ϕ. As a direct consequence of Theorem 2.3, we realise the G-graded Steinberg algebra $A_R(\mathcal{G}_X)$ as the partial skew group ring $C_R(X) \rtimes_\phi G$ (Proposition 3.7). Specialising to the setting of the Leavitt path algebra of a directed graph E, we recover Gonçalves and Royer's result [24, Theorem 3.3], showing that $L_R(E)$ is graded isomorphic to the partial skew group ring via an isomorphism of groupoids (Corollary 4.2). For the case $G = \mathbb{Z}$, we give a condition on the directed graph E so that the Leavitt path algebra of E can be realised as the partial skew group ring of $\mathbb{Z}$ on $C_R(X)$.

Recall that when $\mathcal{G}$ is an ample, Hausdorff groupoid with a continuous cocycle $c : \mathcal{G} \to G$ and $c^{-1}(\varepsilon)$ is strongly effective, there is a one-to-one correspondence between the open invariant subsets of $\mathcal{G}^{(0)}$ and the graded ideals of the Steinberg algebra $A_K(\mathcal{G})$ of $\mathcal{G}$ (see [12, Theorem 5.3]). We observe that the associated groupoid $\mathcal{G}_X$ is an ample, Hausdorff groupoid if X is locally compact Hausdorff with a basis of compact open sets. Applying Proposition 3.7 there is a one-to-one correspondence between the open invariant subsets of $\mathcal{G}_X^{(0)}$ and the graded ideals of the partial skew group ring $C_R(X) \rtimes_\phi G$.

Exel constructed the partial group C^*-algebra [19, Definition 6.4] of a group G. We consider the algebraic version of the construction and realise it as a partial skew group ring associated to a partial action of an abelian group G on an appropriate set.

The paper is organised as follows. In Section 2, we realise the Steinberg algebra of $\mathcal{G}$ as a partial skew inverse semigroup ring. In Section 3, we consider the partial action of G on X. In subsection 3.1, we prove that the G-graded Steinberg algebra $A_R(\mathcal{G}_X)$ is graded isomorphic to the partial skew group ring $C_R(X) \rtimes_\phi G$. In subsection 3.2, we prove the one-to-one correspondence between the open invariant subsets of $\mathcal{G}_X^{(0)}$ and the graded ideals of the partial skew group ring. In subsection 3.3, we give the example of a partial skew group ring arising from an abelian group G. In Section 4, we realise Leavitt path algebras as graded partial skew group ring, where the grading is over a free group. We also describe a graph condition under which the associated Leavitt path algebra can be realised as a $\mathbb{Z}$-graded partial skew group ring.

Historical notes. While preparing this paper, Beuter and Gonçalves posted [7] on arXiv which contains two main theorems: Theorem 3.2 in [7] proves that $C_R(X) \rtimes_\phi G$ is isomorphic to $A_R(\mathcal{G}_X)$ as R-algebras and Theorem 5.2 realises Steinberg algebra of an ample Hausdorff groupoid as a partial skew inverse semigroup ring. Our Theorem 2.3 improve their Theorem 5.2 by showing that there is a graded isomorphism on the level of ideals and that Theorem 3.2 in [7] is direct consequence of Theorem 2.3.

2. Steinberg algebras and partial skew inverse semigroup rings

In this section, we consider a G-graded ample Hausdorff groupoid $\mathcal{G}$, where G is a discrete group and its associated Steinberg algebra $A_R(\mathcal{G})$. The main result of this section (Theorem 2.3) is to realise the G-graded