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# Multifraction reduction IV: Padding and Artin–Tits monoids of sufficiently large type

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ABSTRACT

We investigate the padded version of reduction, an extension of multifraction reduction as defined in arXiv:1606.08991, and connect it both with ordinary reduction and with the so-called Property H. As an application, we show that all Artin–Tits groups of sufficiently large type satisfy some weakened version Conjecture $\mathbf{A}^{\text{padded}}$ of Conjecture **A**, thus showing that the reduction approach is relevant for these groups.



Reduction of multifractions, which was introduced in [2] and [3], is a new approach to the word problem for Artin–Tits groups and, more generally, for groups that are enveloping groups of monoids in which the divisibility relations have weak lattice properties ("gcd-monoids"). It is based on a rewrite system ("$\mathcal{R}$-reduction") that extends the usual free reduction for free groups, as well as the rewrite systems known for Artin–Tits groups of spherical type, and more generally Garside groups. It was proved in [2] that $\mathcal{R}$-reduction is convergent for all Artin–Tits groups of type FC, and in [3] that a certain condition called semi-convergence, weaker than convergence, is sufficient to obtain the decidability of the word problem, leading to the main conjecture ("Conjecture **A**") that $\mathcal{R}$-reduction is semi-convergent for every Artin–Tits monoid.

The aim of the current paper is to exploit the observation that semi-convergence up to Turing-computable padding, a weakening of semi-convergence, is again sufficient to solve the word problem. By *padding*, we mean the insertion of an even number of trivial components at the beginning of a multifraction.

The main results we prove are as follows. First, we have a simple criterion for the word problem:

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**Proposition 1.6.** *If $M$ is a strongly noetherian gcd-monoid with finitely many basic elements, for which $\mathcal{R}$-reduction is semi-convergent up to $f$-padding for some Turing-computable map $f$, then the word problem of $\mathcal{U}(M)$ is decidable.*

Next, we establish a simple connection between the padded version of semi-convergence, the semi-convergence of a variant of $\mathcal{R}$-reduction ("split reduction" or "$\mathcal{S}$-reduction") and Property H of [1,4,8]. When we say below that $(S, R)$ is an lcm presentation for $M$, we mean that every relation $s... = t...$ in $R$ specifies the least common multiple of $s$ and $t$, and similarly on the right; see Definition 1.14.

**Proposition 1.14.** *If $M$ is a gcd-monoid and $(S, R)$ is an lcm presentation for $M$, then the following are equivalent:*

(i) *$\mathcal{R}$-reduction is semi-convergent for $M$ up to padding;*
(ii) *$\mathcal{S}$-reduction is semi-convergent for $M$;*
(iii) *Property* H *is true for $(S, R)$.*

Finally, we consider the specific case of Artin–Tits groups. In view of Proposition 1.6, we propose

**Conjecture $\mathbf{A}^{\text{padded}}$.** *For every Artin–Tits monoid, $\mathcal{R}$-reduction is semi-convergent up to $f$-padding for some Turing-computable map $f$.*

By the results of [2], Conjecture $\mathbf{A}^{\text{padded}}$ is true for every Artin–Tits monoid of type FC. Here we prove:

**Theorem 1.** *Conjecture $\mathbf{A}^{\text{padded}}$ is true for all Artin–Tits monoids of sufficiently large type.*

We recall from [10] that an Artin–Tits group is said to be of *sufficiently large type* if, in any triangle in the associated Coxeter diagram, either no edge has label 2, or all three edges have label 2, or at least one edge has label $\infty$. The result follows directly from the more precise result stated as Proposition 2.1 below, which gives an explicit quadratic upper bound on the padding that is needed. The proof relies on a careful analysis of the techniques of [10] and [8]. With this result, the family of Artin–Tits for which multifraction reduction is relevant is greatly enlarged.

The paper is organised in two sections. The first one is devoted to padded reduction and its variants in a general context of gcd-monoids, and contains a proof of Propositions 1.6 and 1.15. The second section is devoted to the specific case of Artin–Tits monoids of sufficiently large type, with a proof of Theorem 1.

## 1. Padded multifraction reduction

After recalling in Subsection 1.1 the definitions that we need for multifraction reduction and the rewrite system $\mathcal{R}$, we introduce in Subsection 1.2 padded versions of semi-convergence and use them to solve the word problem of the enveloping group. Next, we introduce in Subsection 1.3 a new rewrite system $\mathcal{S}_M$, a variant of $\mathcal{R}$ called split reduction, and we connect its semi-convergence with the padded semi-convergence of $\mathcal{R}$. Finally, we establish the connection with subword reversing and Property H in Subsection 1.4.

### *1.1. Multifraction reduction*

If $M$ is a monoid, we denote by $\mathcal{U}(M)$ the enveloping group of $M$, and by $\iota$ the canonical (not necessarily injective) morphism from $M$ to $\mathcal{U}(M)$. We say that a finite sequence $\underline{a} = (a_1, \dots, a_n)$ of elements of $M$, also called a *multifraction* on $M$ and denoted by $a_1/\cdots/a_n$, *represents* an element $g$ of $\mathcal{U}(M)$ if