Journal of Pure and Applied Algebra 222 (2018) 4126–4150

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Graded Tambara functors

Vigleik Angeltveit [a,*], Anna Marie Bohmann [b,*]

[a] Mathematical Sciences Institute, Australian National University, Canberra, ACT 0200, Australia
[b] Department of Mathematics, Vanderbilt University, 1326 Stevenson Center, Nashville, TN, USA

ARTICLE INFO

Article history:
Received 12 May 2015
Received in revised form 2 February 2018
Available online 2 April 2018
Communicated by S. Iyengar

MSC:
Primary: 55P91; secondary: 55P43; 55N91; 55Q91

ABSTRACT

We define the notion of an $\mathcal{RO}(G)$-graded Tambara functor and prove that any G-spectrum with *norm multiplication* gives rise to such an $\mathcal{RO}(G)$-graded Tambara functor.

1. Introduction

Let G be a finite group. The basic algebraic concept arising in G-equivariant homotopy theory is that of a G*-Mackey functor*. Mackey functors are well studied and their important role in equivariant homotopy theory has been documented since the 1970s—see, e.g., [4] or [13]. For example, the appropriate G-equivariant version of cohomology with coefficients in an abelian group is cohomology with coefficients in a G-Mackey functor. Such a cohomology theory is represented by a G-equivariant Eilenberg–MacLane spectrum.

A commutative multiplication on this type of G-cohomology theory produces a more complicated algebraic structure, called a *Tambara functor*. This type of structure arises from a commutative G-equivariant Eilenberg–MacLane ring spectrum [14]. Compared with Mackey functors, Tambara functors have additional structure. Loosely speaking, a Mackey functor M consists of an abelian group $M(G/H)$ for each subgroup $H \leq G$, together with restriction and transfer maps between these groups that satisfy certain relations. A Tambara functor has an additional type of map, called a “norm map,” relating the groups $M(G/H)$. One can think of restriction as an equivariant version of a diagonal map, transfer as an equivariant version of addition, and norm as an equivariant version of multiplication.

* Corresponding authors.
E-mail addresses: vigleik.angeltveit@anu.edu.au (V. Angeltveit), am.bohmann@vanderbilt.edu (A.M. Bohmann).

https://doi.org/10.1016/j.jpaa.2018.02.023
0022-4049/

The theory of Tambara functors, which was first introduced by Tambara in [12] under the name of TNR-functors, is not as well developed as that of Mackey functors. Brun [3] and Strickland [11] both discuss Tambara functors with an eye towards homotopy theory. Both prove that the zeroth homotopy group of an E_∞-ring spectrum form a Tambara functor. However, neither work considers the algebraic structure present on the homotopy groups in nonzero grading. The main goal of the present work is to understand this structure, which is that of a *graded* Tambara functor. Thus, our results can be thought of as generalizing Strickland's and Brun's work. At the Mackey functor level, the analogous results are well known, but our work requires a refinement of the existing treatment of *graded* Mackey functors. To set the stage for our graded Tambara functors we define the notion of an $\mathcal{RO}(G)$-graded Mackey functor, and prove that any G-spectrum E determines such an $\mathcal{RO}(G)$-graded Mackey functor in Theorem 1.2. Here $\mathcal{RO}(G)$ is a categorification of the real representation ring of G, and our first task is to define $\mathcal{RO}(G)$ carefully.

Additionally, the literature suggests that E has to be a G-equivariant E_∞ ring spectrum in order for its homotopy groups to define a Tambara functor, see e.g. the first paragraph on p. 235 of [2]. This condition seems stronger than necessary, as in the case $G = \{e\}$ a homotopy associative and commutative multiplication clearly suffices to give $\pi_* E$ the structure of a graded commutative ring. We remedy this situation by defining the notion of a *norm multiplication* on a G-spectrum, and prove in Theorem 1.4 that if E has a norm multiplication then its $\mathcal{RO}(G)$-graded homotopy groups constitute a graded Tambara functor.

1.1. Statement of results

Our first contribution is a precise definition of the categorified representation ring. Given a finite G-set X, we make the following definition.

Definition 1.1. Let $\mathcal{RO}(G)(X)$ denote the category $Fun(\mathcal{B}_G X, \widehat{\mathcal{I}}^{\mathrm{op}})$ of functors from the translation category $\mathcal{B}_G X$ to the Grayson–Quillen construction on the category of finite dimensional real inner product spaces. The morphisms are natural transformations.

As a first step we define a category $\mathcal{RO}(G)$ by declaring a morphism from (X, χ) to (Y, γ) to be a pair $(f, \tilde{f})$ where

$$f \colon X \to Y$$

is an isomorphism of G-sets and

$$\tilde{f} \colon \chi \Rightarrow \gamma \circ f$$

is a natural transformation of functors from $\mathcal{B}_G X$ to $\widehat{\mathcal{I}}^{\mathrm{op}}$.

We go on to define a category $\mathcal{RO}(G)^{\mathrm{Mack}}$ by adding restriction and transfer maps to $\mathcal{RO}(G)$, and define an $\mathcal{RO}(G)$-graded Mackey functor to be a functor $\mathcal{RO}(G)^{\mathrm{Mack}} \to Ab$ to the category of abelian groups. The following result is also restated as Theorem 4.11:

Theorem 1.2. *Let E be an orthogonal G-spectrum. Then E determines an $\mathcal{RO}(G)$-graded Mackey functor*

$$\underline{\pi}_\star(E) \colon \mathcal{RO}(G)^{\mathrm{Mack}} \to Ab.$$

Next we define a category $\mathcal{RO}(G)^{\mathrm{Tamb}}$ by also adding norm maps to $\mathcal{RO}(G)$, and define an $\mathcal{RO}(G)$-graded Tambara functor to be a functor $\mathcal{RO}(G)^{\mathrm{Tamb}} \to Set$ landing in the full subcategory of abelian groups. We then pin down the exact amount of multiplicative structure E needs in order for $\underline{\pi}_\star(E)$ to determine an $\mathcal{RO}(G)$-graded Tambara functor.