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Counter-examples to non-noetherian Elkik's approximation theorem

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A R T I C L E I N F O

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A B S T R A C T

Elkik established a remarkable theorem that can be applied for any noetherian henselian ring. For algebraic equations with a formal solution (restricted by some smoothness assumption), this theorem provides a solution adically close to the formal one in the base ring. In this paper, we show that the theorem would fail for some non-noetherian henselian rings. These rings do not satisfy several conditions weaker than noetherianness, such as weak proregularity (due to Grothendieck et al.) of the defining ideal. We describe the resulting pathologies.

1. Introduction

The main goal of this paper is to show that *Elkik's approximation theorem* (Theorem 2.7) would fail in some non-noetherian cases. Elkik's approximation theorem is used for giving affirmative answers to a fundamental question in M. Artin's celebrated work [4]. We first recall it:

Question 1.1 *(cf. [4, Question 1.7]). Let (A, I) be a pair consisting of a ring A and its ideal I. Set $\widehat{A}$ to be the I-adic completion of A. Let $f = (f_1, \ldots, f_m)$ be a polynomial system in $A[X_1, \ldots, X_N]$, and suppose that the equation system $f = 0$ (i.e. $f_1 = 0, \ldots, f_m = 0$) has a solution $\widehat{\boldsymbol{\alpha}} = (\widehat{\alpha}_1, \ldots, \widehat{\alpha}_N) \in \widehat{A}^N$. Let c be a positive integer. Does there exist a solution $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N) \in A^N$ of $f = 0$ such that $\alpha_i \equiv \widehat{\alpha}_i \mod I^c\widehat{A}$ $(i = 1, \ldots, N)$?*

For an important class of henselian pairs, Artin proved that each pair has the following property (called the Artin approximation property): for every equation system, every solution, and every $c > 0$, Question 1.1 has an affirmative answer ([4, Theorem 1.10]). Artin's result has been generalized to a far stronger form below (cf. [2], [13], [14], [15], [16], and [22]): if (A, I) is noetherian[1] (i.e. A is noetherian) and henselian, and the natural morphism $A \to \widehat{A}$ is regular, then (A, I) has the Artin approximation property.

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[1] In [12] and [20], it is shown that some interesting non-noetherian pairs have the Artin approximation property.

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In Elkik's theorem, the equations and the solutions are restricted by some smoothness assumption, which reflects a good lifting property of a henselian pair described in [5, Lemme 2]. By virtue of this, the theorem can be applied for a far broader class of henselian pairs, including all noetherian ones. We denote by $(*)$ the condition imposed on henselian pairs. For (A, I), $(*)$ means either one of the following conditions (cf. §2.2):

($\mathbf{E_a}$) *I is principal, and A has bounded I-torsion (i.e. there exists an integer $l > 0$ such that $I^l A_{I\text{-tor}} = (0)$, cf. §2);*

($\mathbf{E_b}$) *there exists a noetherian pair (A_0, I_0) such that A is flat over A_0 and $I = I_0 A$.*

The basic purpose of this paper is to investigate what part of $(*)$ is essential for Elkik's theorem. The following is our main result, which particularly claims that the bounding condition on $A_{I\text{-tor}}$ is crucial when I is principal, but cannot be substituted for $(*)$.

Main result. *In Elkik's approximation theorem (Theorem 2.7), $(*)$ cannot be weakened to either one of the following conditions:*

($\mathbf{E'_a}$) *I is principal;*

($\mathbf{E''_a}$) *I is finitely generated, and A has bounded I-torsion;*

($\mathbf{E'_b}$) *there exists a pair (A_0, I_0) such that A_0 is noetherian outside I_0 (i.e. the scheme* $\operatorname{Spec} A_0 \setminus V(I_0)$ *is noetherian), A is flat over A_0, and $I = I_0 A$.*

We prove this statement by giving two examples in §3.2. The construction is based on Greco and Salmon's example of non-flat I-adic completion (where $I = tA$). This example gives a negative answer to Question 1.1 (Example 3.1), but it lacks the smoothness assumption that we require. To overcome this, we define an auxiliary equation, and impose suitable relations on A. These relations produce an element $\xi_n \in A$ for every integer $n \geq 1$, such that $t^n \xi_n \neq 0$ but $t^{n+1}\xi_n = 0$ (and then $(*)$ is no longer satisfied). The relation $t^{n+1}\xi_n = 0$ contributes to solving the additional equation in $\widehat{A}$, and the other $t^n \xi_n \neq 0$ is required to conclude the algebraic approximation fails.

On the course of this study, it was found[2] that Elkik's theorem is related to *weak proregularity*, a notion originated in Grothendieck's work[3] [9]. For a pair (A, I), weak proregularity of I allows the derived functors of I-torsion and I-adic completion to behave well (see [17] for details). We note that the weak proregularity condition is a generalization of $(*)$. Moreover, when I is principal, they are equivalent. Thus the main result on ($\mathbf{E'_a}$) signifies the importance of weak proregurarity for Elkik's Theorem in this case. On the other hand, one of the henselian pairs given in §3.2 is defined by a non-principal ideal, which is not weakly proregular. These facts might say that weak proregularity is a key notion for Elkik's algebraic approximation.

The organization of the paper is as follows. In §2, we give some terminology and preliminary results for later use. In §3, we first recall Greco and Salmon's example, and then construct the principal examples to prove the main result. In §4, we check the statements on weak proregularity given in the preceding paragraph.

2. Terminology and preliminaries

Throughout this paper, all rings are assumed to be commutative with unit. By a *pair* we mean a pair (A, I) consisting of a ring A and its ideal I. A *morphism of pairs* $u : (A, I) \to (B, J)$ means a ring homomorphism $u : A \to B$ such that $u^{-1}(J) = I$.

[2] Liran Shaul informed the author (cf. Acknowledgements).

[3] The term "weakly proregular" is due to [1, Correction].

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