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Almost perfect commutative rings

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ABSTRACT

Almost perfect commutative rings R are introduced (as an analogue of Bazzoni and Salce's almost perfect domains) for rings with divisors of zero: they are defined as orders in commutative perfect rings such that the factor rings R/Rr are perfect rings (in the sense of Bass) for all *non-zero-divisors* $r \in R$. It is shown that an almost perfect ring is an extension of a T-nilpotent ideal by a subdirect product of a finite number of almost perfect domains. Noetherian almost perfect rings are exactly the one-dimensional Cohen–Macaulay rings. Several characterizations of almost perfect domains carry over practically without change to almost perfect rings. Examples of almost perfect rings with zero-divisors are abundant.

1. Introduction

About fifteen years ago, Bazzoni and Salce introduced a new type of integral domain that inspired a large number of publications. They called an integral domain R *almost perfect* [5] if every proper factor ring of R was a perfect ring in the sense of Bass [1]. The importance of this concept is shown also by the following incomplete list of properties, each of which characterizes almost perfect domains among integral domains; e.g. see the survey [27]. (For definitions see Section 2.)

(i) Every proper factor ring of R is a perfect ring.
(ii) Flat R-modules are strongly flat.
(iii) Matlis-cotorsion R-modules are Enochs-cotorsion.
(iv) R-modules of weak-dimension ≤ 1 have also projective dimension ≤ 1.
(v) The cotorsion pairs $(\mathcal{P}_1, \mathcal{D})$ and $(\mathcal{F}_1, \mathcal{WI})$ are equal.
(vi) Divisible R-modules are weak-injective.
(vii) h-divisible R-modules are weak-injective.

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(viii) Epimorphic images of weak-injective R-modules are again weak-injective.
(ix) R is h-local and every torsion R-module contains a simple submodule.

Almost perfect domains have been generalized to the non-commutative setting by Facchini and Parolin [10], extending most of the results obtained in [5] (primarily (i) and (ix)) to non-commutative rings. However, possible generalization to commutative rings with divisors of zero seemed to be out of question, since Bazzoni and Salce went on showing that a ring with zero-divisors all of whose proper factor rings were perfect rings was already a perfect ring.

We follow a different approach to generalizations to commutative rings with zero-divisors, that is very natural, and above all leads to an equally interesting class of rings. We pick a property from the above list, and search for all commutative rings that enjoy this property. Our first choice was property (iv).

It did not come as a surprise when we found out that in this way we were led directly to a generalization of almost perfectness that virtually ignored zero-divisors. To be more precise: perfectness was required only *modulo regular ideals*, i.e. ideals containing at least one non-zero-divisor. But if we stop here, then we have not accomplished much: there are useless examples with this property (like rings where all non-zero-divisors are invertible, or rings that have such a ring as a summand), and examples with lots of perfect rings modulo their regular ideals (see e.g. Example 7.8), but with only few or possibly none of the properties listed in (i)–(ix). However, if we keep searching, and concentrate on the more demanding condition (v) (which implicitly includes the requirement that $(\mathcal{P}_1, \mathcal{D})$ is a genuine cotorsion pair), we are then led to an unexpected class of rings: rings that are orders in perfect rings (cf. Theorem 5.3). Accordingly, we are studying rings that are *almost perfect* in the sense that they are orders in commutative perfect rings and their factor rings modulo regular ideals are perfect. A domain that is an almost perfect ring in this sense is then just an almost perfect domain as defined by Bazzoni and Salce [5], since its ring of quotients is a field, which is a perfect ring. It turns out that almost perfect rings are non-noetherian generalizations of one-dimensional Cohen–Macaulay rings (Theorem 4.8).

Sections 3–4 will provide more detailed information about the structure of almost perfect rings. In particular, it will be shown that every almost perfect ring is an extension of a T-nilpotent ideal by a subdirect product of a finite number of almost perfect domains. The equivalence of conditions analogous to those in (i)–(viii) above will be verified for almost perfect rings in Section 6. Most of the proofs are simpler than the corresponding proofs available in the literature in the domain case. The final Section 7 is devoted to examples.

Our results raise several relevant questions on almost perfect rings, some analogous to those already answered in the domain case. We plan to discuss these questions in the near future (the problem on covers and envelopes is solved in a forthcoming paper by the first author [11], and a generalization of almost perfectness to rings of higher Krull dimension is discussed in [15]).

2. Definitions and notations

In this paper, we deal exclusively with commutative rings R with identity. The notation $R^\times$ is used for the set of *regular elements*, i.e. the non-zero-divisors of R. An ideal is called *regular* if it contains a regular element. R-Mod stands for the category of R-modules. Q will denote the (classical) ring of quotients of R, and K the module Q/R. Evidently, Q is a flat R-module and w.d.$K \leq 1$. We use the notations p.d. (projective dimension), i.d. (injective dimension), w.d. (weak dimension), while gl. will indicate 'global' dimension.

An R-module T is *torsion* if for every $x \in T$ there exists an $r \in R^\times$ such that $rx = 0$, i.e. the annihilator Ann x is a regular ideal of R. In any R-module M, the set of elements with regular annihilators form a submodule, the *torsion submodule* $t(M)$ of M, and there is an exact sequence $0 \to t(M) \to M \to$