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Serial factorizations of right ideals ☆

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ABSTRACT

In a Dedekind domain D, every non-zero proper ideal A factors as a product $A = P_1^{t_1} \cdots P_k^{t_k}$ of powers of distinct prime ideals P_i. For a Dedekind domain D, the D-modules $D/P_i^{t_i}$ are uniserial. We extend this property studying suitable factorizations $A = A_1 \dots A_n$ of a right ideal A of an arbitrary ring R as a product of proper right ideals $A_1, \dots, A_n$ with all the modules R/A_i uniserial modules. When such factorizations exist, they are unique up to the order of the factors. Serial factorizations turn out to have connections with the theory of h-local Prüfer domains and that of semirigid commutative GCD domains.

1. Introduction

For any positive integer n, the factorization $n = p_1^{t_1} \cdots p_k^{t_k}$ into powers $p_i^{t_i}$ of distinct primes p_i corresponds to the direct-sum decomposition $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/p_1^{t_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_k^{t_k}\mathbb{Z}$ of the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$ as a direct sum of uniserial $\mathbb{Z}$-modules $\mathbb{Z}/\mathbb{Z}p_i^{t_i}$. The factorization of a non-zero integer into primes is essentially unique, and so is a direct-sum decomposition into uniserial $\mathbb{Z}$-modules, because uniserial modules over a commutative ring have local endomorphism rings [8, Corollary 3.4].

This fact can be generalized to the ideals of any (commutative) Dedekind domain. In a Dedekind domain D, every non-zero proper ideal A factors as a product of powers of prime ideals in a unique way up to the order of the factors. The factorization $A = P_1^{t_1} \cdots P_k^{t_k}$ into powers $P_i^{t_i}$ of distinct prime ideals P_i corresponds to the direct-sum decomposition $D/A = D/P_1^{t_1} \oplus \cdots \oplus D/P_k^{t_k}$ of the D-module D/A as the direct sum of the uniserial D-modules $D/P_i^{t_i}$, and these uniserial modules have local endomorphism rings.

Let R be an arbitrary ring, not necessarily commutative. In this paper, we analyze this situation considering the factorizations $A = A_1 \dots A_n$ of a right ideal A of R as a product of proper right ideals $A_1, \dots, A_n$

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with $A_iA_j = A_jA_i$ for every $i, j = 1, \dots, n$, R/A_i a uniserial module for every $i = 1, \dots, n$ and R/A canonically isomorphic to $R/A_1 \oplus \dots \oplus R/A_n$. We call such a factorization $A_1 \dots A_n$ of A a *serial factorization* of A. The endomorphism ring of a uniserial module is not local, but can have at most two maximal ideals and, correspondingly, the direct-sum decomposition of a finite direct sum of uniserial modules is not unique up to isomorphism, but depends on two permutations of two invariants, called monogeny class and epigeny class (Theorem 2.6). Thus it is natural to expect that the corresponding factorizations $A = A_1 \dots A_n$ of A can be different and depend on two permutations. We show that this is not the case, and that serial decompositions, when they exist, are unique up to a unique permutation of the factors (Theorem 2.8).

We consider the right ideals A of a ring R that have a serial factorization. On the one hand, we show that if a right ideal A of a ring R has a serial factorization $A = A_1 \dots A_n$ and B is any right ideal of R containing A, then B has a serial factorization if and only if either $B \supseteq A_i$ for some index $i = 1, \dots, n$ or B is a two-sided ideal of R, and in this case we describe the serial factorization of B (Theorem 3.1). On the other hand, we prove that if A, B are two similar right ideals of ring R (that is, the right modules R/A and R/B are isomorphic) and A has a serial factorization, then B has a serial factorization and either $A = B$ or the right R-module $R/A \cong R/B$ is uniserial (Theorem 3.4).

We determine several properties of serial factorizations of right ideals, giving a number of examples and showing the analogy between the behavior of ideals with serial factorization and the behavior of factorizations of ideals in (commutative) Dedekind rings. A commutative integral domain has the property that all its non-zero ideals have a serial factorization if and only if it is an h-local Prüfer domain (Proposition 3.11). We find a relation between our theory and the theory of semirigid GCD domains and factorizations of elements of a domain into rigid elements [9]. See Section 4.

In this article, R is an associative ring, not necessarily commutative, with identity $1 \neq 0$. Recall that a right module M_R over a ring R is *uniserial* if its lattice of submodules is linearly ordered, that is, if, for any submodules A, B of M_R, either $A \subseteq B$ or $B \subseteq A$. A module is *serial* if it is a direct sum of uniserial submodules. All serial modules considered in this paper will be serial modules of finite Goldie dimension, that is, finite direct sums of uniserial submodules.

2. Right ideals and their serial factorizations

Let M be a right R-module. A finite set $\{ N_i \mid i \in I \}$ of proper submodules of M is *coindependent* if $N_i + (\bigcap_{j \neq i} N_j) = M$ for every $i \in I$, or, equivalently, if the canonical injective mapping $M/\bigcap_{i\in I} N_i \to \oplus_{i\in I} M/N_i$ is bijective [5, Section 2.8]. Every subfamily of a coindependent finite family of submodules is coindependent.

Lemma 2.1. *Let $A_1, \dots, A_n$ be proper right ideals of a ring R such that $A_iA_j = A_jA_i$ for every $i, j = 1, \dots, n$ and the family $\{A_1, \dots, A_n\}$ is coindependent. Then:*

(1) $A_1 \dots A_n = \bigcap_{i=1}^n A_i$.
(2) *If $n \geq 2$, then each A_i is a two-sided ideal.*

Proof. (1) The proof is by induction on n. The case $n = 1$ is trivial. Suppose the result true for $n - 1$ proper right ideals of R. Let $\{A_1, \dots, A_n\}$ be a coindependent family of proper right ideals of R such that $A_iA_j = A_jA_i$ for every $i, j = 1, \dots, n$. By the inductive hypothesis, $A_1 \dots A_{n-1} = \bigcap_{i=1}^{n-1} A_i$. Let us prove that $A_1 \dots A_n = \bigcap_{i=1}^n A_i$. We have that $A_1 \dots A_n = A_iA_1 \dots \widehat{A_i} \dots A_n \subseteq A_i$ for every i, so that $A_1 \dots A_n \subseteq \bigcap_{i=1}^n A_i$. Conversely,

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