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Coleman automorphisms of finite groups and their minimal normal subgroups

Arne Van Antwerpen [1]

Department of Mathematics, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussel, Belgium

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ABSTRACT

In this paper, we show that all Coleman automorphisms of a finite group with self-central minimal non-trivial characteristic subgroup are inner; therefore the normalizer property holds for these groups. Using our methods we show that the holomorph and wreath product of finite simple groups, among others, have no non-inner Coleman automorphisms. As a further application of our theorems, we provide partial answers to questions raised by M. Hertweck and W. Kimmerle. Furthermore, we characterize the Coleman automorphisms of extensions of a finite nilpotent group by a cyclic p-group. Finally, we note that class-preserving Coleman automorphisms of p-power order of some nilpotent-by-nilpotent groups are inner, extending a result by J. Hai and J. Ge, where p is a prime number.

0. Introduction

Let G be a finite group and let $\mathbb{Z}G$ denote the integral group ring of G. Denote by $Z(U(\mathbb{Z}G))$ the center of the unit group $U(\mathbb{Z}G)$ of $\mathbb{Z}G$ and by $N_{U(\mathbb{Z}G)}(G)$ the normalizer of G in $U(\mathbb{Z}G)$. The well known normalizer problem asks whether $N_{U(\mathbb{Z}G)}(G) = GZ(U(\mathbb{Z}G))$. This problem is posed as question 43 in S.K. Sehgal [24]. The first results for p-groups, nilpotent groups and groups with normal Sylow 2-subgroups are due (respectively) to D. Coleman [2], A. Saksonov [23], S. Jackowski and Z. Marciniak [16] with an important observation by J. Krempa. Further work by, among others, S. Sehgal, M. Parmenter, Y. Li [1], E. Jespers and M. Hertweck [13], M. Hertweck [11,12,9], W. Kimmerle [17] and M. Hertweck and W. Kimmerle [14] enlarged the class of groups with a positive answer greatly. M. Mazur [20,21] showed that the question is closely related to the Isomorphism problem. It poses the question whether an isomorphism between two integral group rings $\mathbb{Z}G \cong \mathbb{Z}H$ implies that the groups are isomorphic $G \cong H$. In 2001 M. Hertweck [10] discovered a counterexample to the isomorphism problem by first constructing a counterexample to the normalizer problem. All counterexamples to the normalizer problem, known to date, are obtained using

E-mail address: arne.van.antwerpen@vub.be.

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M. Hertweck's construction [10]. Hence, it opens the question whether there are still large classes of groups which have a positive answer to the normalizer problem (we say they satisfy the normalizer property). S. Jackowski and Z. Marciniak [16] realized, by Coleman's result, that the normalizer property is closely related to automorphisms of the underlying finite group G. Later, M. Hertweck and W. Kimmerle [14] initiated the study of Coleman automorphisms, which are a generalization of the automorphisms obtained in the study of the normalizer property [16]. Using the alternative characterization of the normalizer problem from [16], it is obvious that finite groups with no non-inner Coleman automorphisms have the normalizer property.

In the first section we give the necessary background on the automorphisms in play.

In the second section we show that the Coleman automorphisms of groups with a self-centralizing minimal non-trivial characteristic subgroup are trivial. Using the same techniques, we show that the holomorph of finite simple groups has no non-inner Coleman automorphisms. Also, we partially answer questions 2 and 3 posed by M. Hertweck and W. Kimmerle in [14]. Moreover, we show that if question 2 is answered positively, then so is question 3. Furthermore, the theorems proven in this section show that certain semi-direct products with simple groups, p-groups or symmetric groups as normal base group have no non-inner Coleman automorphisms. In particular, the recent results of J. Hai and J. Guo in [8] are easy consequences of our theorems.

In the third section, Coleman automorphisms of extensions of nilpotent groups by cyclic p-groups are characterized. As a nice consequence, this provides an easier and different proof of E.C. Dade's result (in [3]) that any finite abelian group can be realized as the group of outer Coleman automorphisms of some metabelian group G. Furthermore, the recent results of Y. Li and Z. Li in [19] are straightforward applications of our characterization. It also shows that the theorems of the previous sections can not be naively adapted to the case of a semi-direct product with a nilpotent group as a normal base group.

In the last section, we note that a recent result of J. Hai and S. Ge (in [18]) can easily be generalized. It is shown that the class-preserving Coleman automorphisms of p-power order of certain nilpotent-by-cyclic groups are inner. As a Corollary we then prove that the class-preserving Coleman automorphisms of p-power order of certain nilpotent-by-nilpotent groups are inner.

1. Preliminary results

In this section we give the definition of the investigated automorphisms. We also include crucial known results on these automorphisms. By $\mathrm{Aut}(G)$ (respectively $\mathrm{Inn}(G)$) we denote the automorphisms (respectively inner automorphisms) of a group G. By $\mathrm{conj}(g)$ we denote the inner conjugation $x \mapsto g^{-1}xg$ on G. The group of outer automorphisms is $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$. The set consisting of the prime divisors of the order of a finite group G is denoted by $\pi(G)$. The cyclic group of order n will be denoted C_n for a positive integer n. The commutator of two elements $x, y \in G$ is denoted $[x, y]$.

We begin with the definition of Coleman automorphism, which was introduced by M. Hertweck and W. Kimmerle in [14].

Definition 1.1. Let G be a finite group and $\sigma \in \mathrm{Aut}(G)$. If for any prime p dividing the order of G and any Sylow p-subgroup P of G, there exists a $g \in G$ such that $\sigma|_P = \mathrm{conj}(g)|_P$, then σ is said to be a Coleman automorphism.

The group of Coleman automorphisms is denoted $\mathrm{Aut}_{col}(G)$ and its image in $\mathrm{Out}(G)$ is denoted $\mathrm{Out}_{col}(G) = \mathrm{Aut}_{col}(G)/\mathrm{Inn}(G)$.

Definition 1.2. Let G be a finite group and $\sigma \in \mathrm{Aut}(G)$. If the conjugacy classes of G are invariant under σ, i.e. for any $g \in G$ there exists an $x \in G$ such that $\sigma(g) = x^{-1}gx$, then σ is called a class-preserving automorphism.

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