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Selfinjective algebras without short cycles of indecomposable modules

Alicja Jaworska-Pastuszak*, Andrzej Skowroński

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, Poland

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ABSTRACT

We describe the structure of finite dimensional selfinjective algebras over an arbitrary field without short cycles of indecomposable modules.

0. Introduction and the main result

Throughout the paper, by an algebra we mean a basic indecomposable finite dimensional associative K-algebra with identity over a field K. For an algebra A, we denote by $\operatorname{mod} A$ the category of finite-dimensional right A-modules, by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ formed by the indecomposable modules, by Γ_A the Auslander–Reiten quiver of A, and by τ_A and τ_A^{-1} the Auslander–Reiten translations DTr and TrD, respectively. We do not distinguish between a module in $\operatorname{ind} A$ and the vertex of Γ_A corresponding to it. An algebra A is of finite representation type if the category $\operatorname{ind} A$ admits only a finite number of pairwise nonisomorphic modules. It is well known that a hereditary algebra A is of finite representation type if and only if A is of Dynkin type, that is, the valued quiver Q_A of A is a Dynkin quiver of type $\mathbb{A}_n$ $(n \geqslant 1)$, $\mathbb{B}_n$ $(n \geqslant 2)$, $\mathbb{C}_n$ $(n \geqslant 3)$, $\mathbb{D}_n$ $(n \geqslant 4)$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, $\mathbb{F}_4$ and $\mathbb{G}_2$ (see [7], [8], [9]). A distinguished class of algebras of finite representation type is formed by the tilted algebras of Dynkin type, that is, the algebras of the form $\operatorname{End}_H(T)$ for a hereditary algebra H of Dynkin type and a (multiplicity-free) tilting module T in $\operatorname{mod} H$. An algebra A is called selfinjective if A_A is an injective module, or equivalently, the projective modules in $\operatorname{mod} A$ are injective. For a selfinjective algebra A, we denote by Γ_A^s the stable Auslander–Reiten quiver of A, obtained from Γ_A by removing the projective modules and the arrows attached to them.

* Corresponding author.
E-mail addresses: jaworska@mat.uni.torun.pl (A. Jaworska-Pastuszak), skowron@mat.uni.torun.pl (A. Skowroński).

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We are concerned with the problem of describing the isomorphism classes of selfinjective algebras of finite representation type. For K algebraically closed, the problem was solved in the 1980's by C. Riedtmann (see [5], [17], [18], [19]) via the combinatorial classification of the Auslander–Reiten quivers of selfinjective algebras of finite representation type. Equivalently, Riedtmann's classification can be presented as follows (see [20, Section 3]): a nonsimple selfinjective algebra A over an algebraically closed field K is of finite representation type if and only if A is a socle (geometric) deformation of an orbit algebra $\widehat{B}/G$, where $\widehat{B}$ is the repetitive category of a tilted algebra B of Dynkin type $\mathbb{A}_n$ $(n \geqslant 1)$, $\mathbb{D}_n$ $(n \geqslant 4)$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, and G is an admissible infinite cyclic group of automorphisms of $\widehat{B}$. For an arbitrary field K, the problem seems to be difficult (see [3], [4], [12], [25] for some results in this direction and [26, Section 12] for related open problems). An important known result towards solution of this general problem is the description of the stable Auslander–Reiten quiver Γ^s_A of a selfinjective algebra A of finite representation type established by C. Riedtmann [17] and G. Todorov [30] (see also [27, Section IV.15]): Γ^s_A is isomorphic to the orbit quiver $\mathbb{Z}\Delta/G$, where Δ is a Dynkin quiver of type $\mathbb{A}_n$ $(n \geqslant 1)$, $\mathbb{B}_n$ $(n \geqslant 2)$, $\mathbb{C}_n$ $(n \geqslant 3)$, $\mathbb{D}_n$ $(n \geqslant 4)$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, $\mathbb{F}_4$ or $\mathbb{G}_2$, and G is an admissible infinite cyclic group of automorphisms of the translation quiver $\mathbb{Z}\Delta$. Therefore, we may associate to a selfinjective algebra A of finite representation type a Dynkin graph $\Delta(A)$, called the Dynkin type of A, such that $\Gamma^s_A = \mathbb{Z}\Delta/G$ for a quiver Δ having $\Delta(A)$ as underlying graph. We also mention that, for a tilted algebra B of Dynkin type Δ, the orbit algebras $\widehat{B}/G$ are selfinjective algebras of finite representation type whose Dynkin type is the underlying graph of Δ.

Following [16], a *short cycle* in the module category $\operatorname{mod} A$ of an algebra A is a sequence

$$X \xrightarrow{f} Y \xrightarrow{g} X$$

of two nonzero nonisomorphisms between modules X and Y in $\operatorname{ind} A$. It has been proved in [16, Corollary 2.2] that if M is a module in $\operatorname{ind} A$ which does not lie on a short cycle then M is uniquely determined (up to isomorphism) by its image $[M]$ in the Grothendieck group $K_0(A)$. Moreover, by a result of Happel and Liu [11, Theorem] every algebra A having no short cycles in $\operatorname{mod} A$ is of finite representation type.

The following theorem is the main result of the paper.

Theorem. *Let A be a selfinjective algebra over a field K. The following statements are equivalent.*

(i) $\operatorname{mod} A$ *has no short cycles.*
(ii) *A is isomorphic to an orbit algebra $\widehat{B}/(\varphi\nu^2_{\widehat{B}})$, where B is a tilted algebra of Dynkin type over K, $\nu_{\widehat{B}}$ is the Nakayama automorphism of $\widehat{B}$, and φ is a strictly positive automorphism of $\widehat{B}$.*

The paper is organized as follows. In Section 1 we introduce the orbit algebras of repetitive categories. Section 2 is devoted to presenting basic results from the theory of selfinjective algebras with deforming ideals, playing the fundamental role in the proof of the main theorem. In Section 3 we discuss properties of stable slices of Auslander–Reiten quivers of selfinjective algebras of finite type, essential for further considerations. In Section 4 we describe the selfinjective Nakayama algebras without short cycles of indecomposable modules. In Section 5 we prove the Theorem for the selfinjective algebras of Dynkin type. In the final Section 6 we complete the proof of the Theorem for arbitrary selfinjective algebras.

For basic background on the representation theory applied in this paper we refer to [1], [2], [27] and [28].

1. Orbit algebras of repetitive categories

Let B be an algebra and $1_B = e_1 + \cdots + e_n$ a decomposition of the identity of B into a sum of pairwise orthogonal primitive idempotents. We associate to B a selfinjective locally bounded K-category $\widehat{B}$, called