ARTICLE IN PRESS JPAA:5821

Journal of Pure and Applied Algebra ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Singular rational curves with points of nearly-maximal weight

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ARTICLE INFO

Article history:
Received 14 June 2017
Received in revised form 6 October 2017
Available online xxxx
Communicated by G.G. Smith

MSC:
Primary 14H20; 14H45; 14H51; 20Mxx

ABSTRACT

In this article we study rational curves with a unique unibranch genus-g singularity, which is of κ-*hyperelliptic* type in the sense of [27]; we focus on the cases $\kappa = 0$ and $\kappa = 1$, in which the semigroup associated to the singularity is of (sub)maximal weight. We obtain a partial classification of these curves according to the linear series they support, the scrolls on which they lie, and their gonality.

0. Introduction

Rational curves have long played an essential rôle in the classification of complex algebraic varieties. Even when the target variety is $\mathbb{P}^n$, the problem of classifying *singular* rational curves is surprisingly subtle, and most works to date have concentrated on the case $n = 2$; see for example [8,10,13,15,18,19,21,26]. In this paper we explore the classification problem for rational curves with a unibranch singularity P (and which are smooth away from P).

Semigroups of unibranch singularities naturally form a tree, whose vertices are indexed by their minimal generating sets. The asymptotic structure of the tree's infinite leaves is essentially prescribed by the *weights* of the underlying semigroups. This leads us naturally to a reconsideration of how these weights should be specified in the first place. In this line of questioning, we are guided by an obvious analogy between semigroups of singular points and semigroups of Weierstrass points of linear series on smooth curves.

For a smooth curve, the Weierstrass semigroup is given either by pole orders of meromorphic functions or of vanishing orders of regular differentials, and via Serre duality each formulation is equivalent. For a

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https://doi.org/10.1016/j.jpaa.2017.12.017
0022-4049/

Please cite this article in press as: E. Cotterill et al., Singular rational curves with points of nearly-maximal weight, J. Pure Appl. Algebra (2017), https://doi.org/10.1016/j.jpaa.2017.12.017

singular curve, however, using pole orders or differential orders of vanishing produces two distinct notions of weight. The main premise of Section 0.1 of this paper is that using a notion of weight based on differential orders of vanishing, as in [14] (which, however, makes no mention of semigroups), confers certain advantages.

Torres [27] has given an asymptotic classification of numerical semigroups according to the notion of weight coming from meromorphic functions. Our Theorem 1.4 (whose proof strongly uses Torres' results) gives a slightly more precise characterization, using a notion of weight based on differentials, for *bielliptic* singularities: these are shown to be precisely those nonhyperelliptic singularities of maximal weight. It is natural to speculate that Torres' general classification result for *κ-hyperelliptic* singularities may be refined using our alternative version of weight. We make this quantitatively precise in Conjecture 1.6, and give some evidence for the conjecture in Theorem 1.7.

The analogy with Weierstrass points also leads one naturally to wonder how the stratification of singular rational curves according to *gonality* interacts with the stratification according to value semigroups (indeed, one of the primary motivations behind Torres' theory was to clarify that causal relationship for smooth curves); this is the focus of Section 2. In Theorem 2.2 we give a simple upper bound for the gonality of an integral projective curve as a function of the arithmetic genus, and characterize when the bound is sharp.

The study of (gonality) of linear series on irreducible singular curves is more delicate than the classical theory for smooth curves. It is both useful and necessary from our point of view to authorize *non-removable* base points, in the sense of [7]. For example, it is well-known that any trigonal smooth curve lies on a surface scroll. Stöhr and Rosa showed in [22] that the assertion remains true for *Gorenstein* curves, with the caveats that the scroll might be a cone; and that the g^1_3 might have non-removable base points. More generally, the fact that any smooth d-gonal curve embeds in a $(d-1)$-fold scroll is due to Bertini; see, e.g. [24, Thm. 2.5]. Theorem 2.5 establishes that rational curves with a single unibranch singularity lie on scrolls of a certain (co)dimension that is computable from their parametrizations. When the singularity is *bielliptic*, we also obtain (sufficient) conditions for the curve to admit pencils with a non-removable base point. We also give an intrinsic characterization (in terms of k) of those rational curves that carry base-point-free g^1_k's in Lemma 2.4.

Finally, in Section 3, we study rational curves with singularities of maximal and submaximal weight. Theorem 3.1 characterizes hyperelliptic (singular) curves. It should be compared against the well-known characterization of hyperelliptic smooth curves as those for which 2 belongs to the Weierstrass semigroup in a point. Within our category of singular rational curves (with a single unibranch singularity) the analogous characterization fails: hyperelliptic singular curves have hyperelliptic singularities, i.e. singularities for which 2 belongs to the corresponding numerical semigroup, but not vice versa. Accordingly, we attempt to characterize those curves with hyperelliptic singularities that are not (globally) hyperelliptic, i.e. that admit a degree-2 morphism to $\mathbb{P}^1$. Proposition 3.2 gives a complete resolution in the first nontrivial case of genus 3. We obtain some analogous results for bielliptic singular curves and curves with bielliptic singularities in Theorem 3.3 and Remark 3.4, respectively.

The connection between gonality, embeddings in scrolls, and rational curves with κ-hyperelliptic singularities is a theme to which we intend to return in the future.

0.1. Conventions for rational curves and their singularities

We work over $\mathbb{C}$. By *rational curve* we always mean a projective curve of geometric genus zero. A *numerical semigroup* is a subsemigroup $\mathrm{S} \subset \mathbb{N}$ of the natural numbers with finite complement G_{S}; the *genus* $g = g(\mathrm{S})$ is equal to the cardinality of G_{S}. The *genus* of a value semigroup of a singularity encodes the contribution of that singularity to the arithmetic genus of the underlying projective curve.

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